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Short communication

Stresses in thick-walled FGM cylinders with exponentially-varying properties

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Abstract

Power series solutions for stresses and displacements in functionally-graded cylindrical vessels subjected to internal pressure alone are obtained using the infinitesimal theory of elasticity. The material is assumed to be isotropic with constant Poisson's ratio and exponentially-varying elastic modulus through the thickness. Stress distributions depending on an inhomogeneity constant are calculated and presented in the form of graphs. The inhomogeneity constant which includes continuously varying volume fraction of the constituents is empirically determined. The values used in this study are arbitrarily chosen to demonstrate the effect of inhomogeneity on stress distribution.

Keywords: FGM; Pressure vessel; Elasticity; Axisymmetry

1. Introduction

Functionally graded materials (FGMs) have attracted much interest primarily as heat-shielding materials. The possibility of tailoring the desired thermomechanical properties holds enormous application potential for FGMs. Aside from the thermal barrier coatings, some of the potential applications of FGMs include their use as interfacial zones to improve the bonding strength and to reduce residual stresses in bonded dissimilar materials and as wear-resistant layers such as gears, cams, ball and roller bearings and machine tools (Erdogan [1]). Most of the studies conducted on FGMs are confined to the analysis of thermal stress and deformation (see, e.g., Wetherhold et al. [2], Takezono et al. [3], Zhang et al. [4], Obata and Noda [5]). The works concerning the stress analysis of cylindrical and spherical structural elements involve finite elements and other numerical techniques due to the nature of functions chosen to describe the inhomogeneous properties (Fukui and Yamanaka, [6] Loy et al. [7], Salzar [8]).

Developing sufficiently general methods for solving specific boundary value problems in solid mechanics involving inhomogeneous media has always been difficult. Because of this difficulty, all existing treatments dealing with the mechanics of inhomogeneous solids are based on a simple function representing material inhomogeneity. For example, in the half-plane elasticity problems considered by Kassir and Chauprasert [9] and Kassir [10] it is assumed that the shear modulus is a power function of the depth coordinate of the form $\mu(y) = \mu_0 y^m$ and the Poisson's ratio ν is constant. Modeling of density and stiffness by the same power-law are proposed by Bert and Niedenfuhr [11], Reddy and Srinath [12] and Gurushankar [13]. The functionally gradient material considered by Loy et al. [7] is composed of stainless steel and nickel where the volume fractions follow a power-law distribution. Closed-form solutions are obtained by Tutuncu and Ozturk [14] for cylindrical and spherical vessels with variable elastic properties obeying a simple power law through the wall thickness which resulted in simple Euler–Cauchy equations whose solutions were readily available. A similar work was also published by Horgan and Chan [15] where it was noted that increasing the positive exponent of the radial coordinate provided a stress shielding effect whereas decreasing it created stress amplification. Three-dimensional solutions for FGM plates are obtained numerically by Reddy and Cheng [16] using the transfer matrix method. The overall material properties were calculated from the constituent

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properties by the well-known Mori–Tanaka method. On the stress analysis of piezoelectric plates where the piezoelectric properties are functionally graded the work by Lim and He [17] presents exact solutions. Similar works on piezoelectric FGM plates are also presented by Liew et al. [18,19]. Free vibration analysis of such plates is performed by Lim et al. [20] where the transfer matrix method combined with the asymptotic expansion method is used. The parametric resonance of FGM rectangular plates is studied by Ng et al. [21] where Hamilton's principle and Bolotin's method are used to determine the instability regions.

The present paper aims to present stress and displacement solutions in thick-walled cylinders subjected to internal pressure only. The material is assumed to be isotropic with exponentially-varying elastic modulus through the thickness as $E(r) = E_0 e^{\beta r}$ yielding governing equations solutions of which are not readily available and can only be obtained in the form of power series by employing the lengthy process of Frobenius method. When the functional dependence is assumed for both the elastic modulus and Poisson's ratio a simple tractable solution cannot be obtained necessitating the employment of numerical and perturbation techniques. For the sake of simplicity the insignificant influence of the variation in Poisson's ratio on stresses is neglected and a constant Poisson's ratio ν is assumed throughout the thickness as it is done in numerous works in the literature such as those by Erdogan [1], Horgan and Chan [15], Chen and Erdogan [22] and Jabbari et al. [23]. Various β values are used to demonstrate the effect of inhomogeneity on the stress distribution. The arbitrary values used in this study for the inhomogeneity constant β do not necessarily represent a certain material.

2. Analysis

The stress distribution in thick-walled cylindrical pressure vessels will be calculated. Elastic modulus for the isotropic material is assumed to vary as

$$E(r) = E_0 e^{\beta r}. \tag{1}$$

Employing the plain-strain assumption and axisymmetry, the strain–displacement and constitutive equations are

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}, \qquad \varepsilon_\theta = \frac{u}{r}, \qquad \gamma_{r\theta} = 0 \tag{2}$$

$$\begin{aligned} \sigma_r &= C_{11}\varepsilon_r + C_{12}\varepsilon_\theta \\ \sigma_\theta &= C_{12}\varepsilon_\mathrm{r} + C_{11}\varepsilon_\theta \end{aligned} \tag{3a,b}$$

where, with ν_0 being the Poisson's ratio,

$$\begin{aligned} C_{11} &= C_{11}^0 e^{\beta \mathrm{r}} = \left(\frac{E_0(1-\nu_0)}{(1+\nu_0)(1-2\nu_0)}\right) e^{\beta r} \text{ ve } C_{12} \\ &= C_{12}^0 e^{\beta \mathrm{r}} = \left(\frac{E_0 \nu_0}{(1+\nu_0)(1-2\nu_0)}\right) e^{\beta r}. \end{aligned}$$

The only nontrivial equilibrium equation is

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{4}$$

Using Eqs. (1)–(3), the governing equation of radial displacement becomes

$$r^2 u'' + r(1 + r\beta)u' + (\nu\beta r - 1)u = 0 \tag{5}$$

where $\nu = \frac{C_{12}}{C_{11}} = \frac{\nu_0}{1-\nu_0}$.

Eq. (5) can be solved by Frobenius Method with the solution in the form

$$u(r) = \sum_{k=0}^{\infty} a_k r^{k+s}. \tag{6}$$

Substituting in Eq. (5) gives the recurrence formula

$$a_k = -\frac{(k+s-1)+\nu}{(k+s+1)(k+s-1)}\beta a_{k-1} \tag{7}$$

and the indicial equation $(s-1)(s+1) = 0$. Since the roots of the indicial equation differ by an integer ($s_1 = 1, s_2 = -1$) only one of the solutions is in the form of Eq. (6). Expansion of the recurrence formula for $k = 1, 2, 3, \ldots$ gives the coefficients a_k in terms of a_0 and Gamma functions as:

$$a_k = \frac{(-1)^k \Gamma(2+s)^2 \Gamma(k+s+\nu)}{\Gamma(s+\nu)\Gamma(k+s)\Gamma(k+2+s)(1+s)s}\beta^k a_0. \tag{8}$$

For the first root $s = 1$, taking the nonzero arbitrary constant $a_0 = 1$ the recurrence relation takes the following form:

$$a_k = \frac{2(-1)^k}{k!(k+2)!}\frac{\Gamma(k+1+\nu)}{\Gamma(1+\nu)}\beta^k. \tag{9}$$

Here, for an integer k, the property $\Gamma(k+1) = k!$ has been used. The first solution is given as

$$u_1 = \sum_{k=0}^{\infty} a_k r^{k+1}. \tag{10}$$

The second solution for $s = -1$ will be of the form

$$\begin{aligned} u_2 &= \sum_{k=0}^{\infty} \{(s+1)a_k(s) r^{k+s}\}_{s=-1} \log r \\ &\quad + \sum_{k=0}^{\infty} \left\{\frac{\mathrm{d}}{\mathrm{d}s}[(s+1)a_k(s)]\right\}_{s=-1} r^{k-1}. \end{aligned} \tag{11}$$

The multiplier of the logarithmic term is expanded first as

$$\begin{aligned} &\sum_{k=0}^{\infty} \{(s+1)a_k(s) r^{k+s}\}_{s=-1} \\ &= a_0 \sum_{k=0}^{\infty} \left[\frac{(s+1)(-1)^k \Gamma(2+s)^2 \Gamma(k+s+\nu)}{\Gamma(s+\nu)\Gamma(k+s)\Gamma(k+2+s)(1+s)s}\beta^k r^{k+s}\right]_{s=-1} \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^{k-1}\Gamma(k-1+\nu)}{\Gamma(\nu-1)\Gamma(k-1)\Gamma(k+1)}\beta^k r^{k-1} \\ &= a_0 \beta^2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\Gamma(k+1+\nu)}{\Gamma(\nu-1)k!(k+2)!}\beta^k r^{k+1}. \end{aligned} \tag{12}$$

It should be noted that since $\Gamma(0) = \infty$ and $\Gamma(-1) = \infty$, the summation should start from $k = 2$. Subsequently, the indices are changed as $k \to k+2$ to obtain the final form.