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An Approximation to the Probability Normal Distribution and its Inverse

Una aproximación a la distribución de probabilidad normal y su inversa

Alamilla-López Jorge Luis
Instituto Mexicano del Petróleo
Dirección de Investigación en Transformación de Hidrocarburos
E-mail: jalamill@imp.mx

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Abstract

Mathematical functions are used to compute Normal probabilities, which absolute errors are small; however their large relative errors make them unsuitable to compute structural failure probabilities or to compute the menace curves of natural hazards. In this work new mathematical functions are proposed to compute Normal probabilities and their inverses in an easy and accurate way. These functions are valid over a wide range of random variable and are useful in applications where computational speed and efficiency are required. In addition, these functions have the advantage that the numerical correspondence between the random value $X = x$ and its Normal probability $\Phi(-x)$ is bijective.

Keywords:

- normal distribution
- inverse
- absolute error
- relative error

Resumen

Se utilizan funciones matemáticas para calcular probabilidades normales, cuyos errores absolutos son pequeños; sin embargo, sus grandes errores relativos las hacen inadecuadas para calcular probabilidades de falla de estructuras o para calcular curvas de amenaza de peligros naturales. En este trabajo se proponen nuevas funciones matemáticas para calcular probabilidades normales y sus inversas, de manera fácil y precisa. Estas funciones son válidas en un intervalo amplio de valores que puede tomar la variable aleatoria y son útiles en aplicaciones donde se requiere velocidad y eficiencia computacional. Además, estas funciones tienen la ventaja de que la correspondencia numérica entre el valor aleatorio $X = x$ y su probabilidad Normal $\Phi(-x)$ es biyectiva.

Descriptores:

- *distribución normal*
- *inversa*
- *error absoluto*
- *error relativo*

Introduction

The normal distribution function Φ (-x) is used in engineering probabilistic applications to compute reliabilities of structural systems such as buildings, offshore platforms, pipelines, tanks, and bridges, among others. These applications require accurate estimations of small probabilities, which are associated with relatively large values. In addition, these probabilities have to be easily invertible. The disadvantage of not having a closed analytical form of estimation Φ (-x) has been overcome using mathematical approximations. Summaries of these mathematical approximations are given in (Abramowitz & Stegun, 1972; Patel & Read, 1996), which are useful to compute probabilities or their inverses, but not both. The errors of these approximations are specified in terms of their maximum absolute error. The absolute errors of the approximations are small but their relative errors are significant, which becomes important in the tail of probability distribution. In the present work, this inconvenience is shown for the best mathematical function reported in (Abramowitz & Stegun, 1972), the one with minimum absolute error among the available approximations. Also, other approximations commonly used to compute failure probabilities are revised and new mathematical expressions with no such inconvenience are proposed. These new functions are useful to compute Normal probabilities and their inverses in a relatively easy and fast way, with good accuracy. Furthermore, they have the advantages of being valid over a wide range of the random variable, which makes them useful in engineering applications.

Reference framework

Herein, some of the mathematical functions typically used in engineering applications are reviewed. The number of functions analyzed is not exhaustive; the author only discusses some approaches that in his opinion are representative for estimating probabilities and/or their inverses in structural and mechanical engineering. The analysis is focused on identifying the accuracy and scope of these functions, and showing the inconveniences mentioned in the previous section.

In general, the normal probability distribution function Φ (·) can be described in terms of its probability density function φ (·) as follows

$$\Phi(-x) = \psi(x)\ \varphi(x) \tag{1}$$

The function ψ(·) relates the normal distribution function with its derivative, the probability density function φ (x). It can also be seen as the one that transforms the probability density function in its cumulative. Herein, to obtain values of ψ(·) for large values of x, the first and second derivatives ψ(·) are taken: ψ', ψ'' respectively, hence it gives the linear second order differential equation

$$\psi'' - x\psi' - \psi = 0 \tag{2}$$

This differential equation was solved numerically for the initial conditions ψ (x = 0) = $\sqrt{2\pi}$ / 2 and ψ' (x = 0) = –1. Such a solution is taken as reference for ψ (·). As shown in Figure 1, the asymptotic approximation to describe ψ (x), given by the expression (3), gives good results for large x values.

$$\psi_A(x) = \frac{1}{x} - \frac{1}{x^3} + \frac{1.3}{x^5} - \frac{1.3.5}{x^7} + \cdots \tag{3}$$

For x values close to zero, Eq. (3), diverges from the corresponding numerical function. Although the asymptotic approximation of this function can be used to calculate probabilities associated with large values of x, it has the disadvantage of not being easily invertible.

Two functions to describe ψ (x), that provide good approximation over a wide range of x values, are those proposed by Hastings (Abramowitz and Stegun, 1972; (Rosenblueth, 1985). The Hastings's approach is a classic function, probably the most used to compute Normal probabilities, because it has the minimum absolute error in (Abramowitz and Stegun, 1972), and can be expressed as follows

$$\psi_H(x) = \sum_{j=1}^{5} a_j(1 + a_0\ x)^{-j} \tag{4}$$

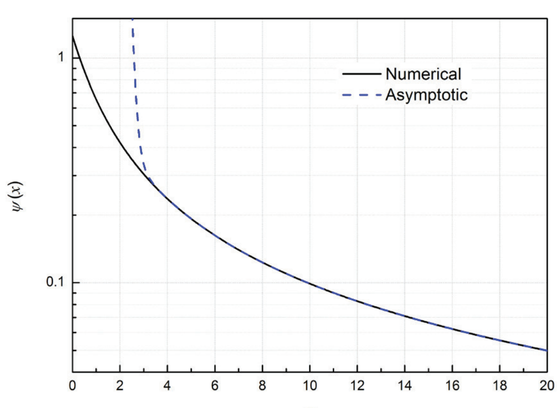

Figure 1. Numerical and asymptotic approach of function ψ (·)