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On constitutive models of finite elasticity with possible zero apparent Poisson's ratio

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ABSTRACT

The idea in this paper is to build a class of constitutive equations for highly compressible isotropic materials that, among others, are capable to describe a zero apparent Poisson's ratio in the whole finite strain range, not only for moderate straining. This remarkable property is, for instance, observed in many soft materials with micro-structures such as sponges and polymeric foams with high porosities. It would then be suitable to describe their behavior within a macroscopic modeling framework. More specifically, herein by means of elementary considerations, we deduce adequate forms of strain-energy functions that are a priori decomposed into purely volumetric and volume-preserving parts. A class of compressible hyperelastic materials of the general Odgen type is obtained. It can consequently be specialized, for instance, to neo-Hookean, Mooney–Rivlin, and Varga's model types as well. Furthermore, for the elastic parameters, a connection with the limiting case of linear elasticity is made whenever possible, in particular with the classical Poisson's ratio, and with the bulk to shear moduli ratio.

1. Introduction

In isotropic linear elasticity theory, the very important Poisson's ratio, known as ν, is determined from kinematical measurements alone. For example, in a simple tension experiment, it is defined as the ratio of the lateral contractive engineering strain to the extensional one. However, in the finite strain range, Poisson's ratio plays a less important role, and only ad-hoc definitions can be adopted speaking then of the Poisson function, see for example Beatty and Stalnaker (1986) for some useful definitions. To avoid any confusion in large deformation, herein we prefer to speak of apparent Poisson's ratio. When a material is uniaxially loaded along a given direction, by apparent Poisson's ratio we mean the ratio between the deformation transverse to the loading and the one along the loading direction. Hence, if the material experiences zero apparent Poisson's ratio, this simply means that there is no contraction nor extension in directions transverse to that of the loading direction. This property can at least be observed in many soft materials, mostly with micro-structures, such as sponges and polymeric foams with high open porosities. It would then be suitable to describe their macroscopic behavior within a continuum modeling framework.

The aim of this paper is precisely to build a class of hyperelastic constitutive models that are capable to describe the zero apparent Poisson's property when adequate sets of material parameters are chosen. As the materials we consider are compressible by essence, we a priori adopt an additive split of the strain-energy function into a purely volumetric part and a volume-preserving part. More specifically, we start the analysis by choosing a known form for the later part, here a form written in terms of the principal stretches due to Ogden (1997), and the challenge is then to find an adequate form for the volumetric part of the strain-energy function. The analysis is conducted with the help of a model problem in simple tension/compression. It is found that the resulting constitutive modeling is remarkably simple, and the connection with the limiting case of linear elasticity is straightforward. Very few models exist in the literature that describe the same properties. However, notice that strain energy functions of the Hencky type based on the logarithmic stretches have been proposed recently, see for example Neff et al. (2015).

An outline of the remainder of this paper is as follows. In Section 2, we define the simple model problem used during the analysis together with the basic governing equations needed to solve it. Then, in Section 3, the analysis starts with the compressible model of the $N = 1$-Ogden type where the desired zero Poisson's ratio property is among others reached and deeply investigated. Next, in Section 4, the class of constitutive models is extended to the complete model due to Ogden. Finally, conclusions are drawn in Section 5.

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2. General form and sample example

Since the solids we consider are (highly) compressible, it is beneficial to locally split the deformation gradient $\boldsymbol{F}$ into a volumetric part, that depends on the Jacobian $J = \det \boldsymbol{F} > 0$, and an isochoric part that depends on the modified deformation gradient $J^{-1/3}\boldsymbol{F}$, as originally proposed by Flory (1961), and successfully applied later on in finite strain elasticity, e.g. see Holzapfel (2000); Lubliner (1985); Simo and Hughes (1998) among many others. We then choose to write

$$\psi(\boldsymbol{C}) = \psi_{\text{vol}}(J) + \psi_{\text{iso}}(J^{-\frac{2}{3}}\boldsymbol{C}), \tag{1}$$

for the strain-energy function ψ that, for objectivity reasons, depends on $\boldsymbol{F}$ only through the right Cauchy–Green tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ which, otherwise, is also a strain measure. The notation $(\cdot)^T$ is used for the transpose operator of a second-order tensor. The first term ψ_{vol} in (1) is related to the volumetric part of the response while the isochoric term, ψ_{iso}, is related to the volume-preserving part. For the stress response, the Cauchy stress tensor, herein denoted by $\boldsymbol{\sigma}$, is given by

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\Big(2\frac{\partial\psi}{\partial\boldsymbol{C}}\Big)\boldsymbol{F}^T \equiv \underbrace{\frac{\partial\psi_{\text{vol}}}{\partial J}\boldsymbol{1}}_{=\boldsymbol{\sigma}_{\text{vol}}} + \underbrace{2J^{-1}\boldsymbol{F}\frac{\partial\psi_{\text{iso}}}{\partial\boldsymbol{C}}\boldsymbol{F}^T}_{=\boldsymbol{\sigma}_{\text{iso}}}, \tag{2}$$

where $\boldsymbol{1}$ is the second-order identity tensor. For a solid of actual configuration $\mathcal{B}_t$, the Cauchy stress tensor is the one that is used in the spatial form of the balance equation given in quasi-statics by

$$\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0} \quad \text{in } \mathcal{B}_t, \quad \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{t} \quad \text{on } \partial\mathcal{B}_t, \tag{3}$$

where $\operatorname{div}(\cdot)$ stands for the spatial divergence operator, and $\boldsymbol{t}$ is the Cauchy traction vector on the boundary $\partial\mathcal{B}_t$ of unit normal $\boldsymbol{n}$. Here and henceforth, the volumetric forces will be neglected for the sake of simplicity.

A model problem will be used throughout the paper that consists of a cylindrical sample uniformly loaded along its axis that is aligned with the fixed direction $\vec{e}_3$, and free laterally. The loading can be tension or compression. However, for compressive loadings, we suppose that the height to diameter ratio is low enough so as to preclude buckling instabilities, a topic that is out of the scope of this paper. The solution of this problem is simple with a homogeneous distribution of the stress tensor of the form

$$\boldsymbol{\sigma} = \sigma_{33}\,\vec{e}_3 \otimes \vec{e}_3, \tag{4}$$

where the notation $\otimes$ stands for the tensor product. The cylinder undergoes a homogeneous transformation as well with identical transverse stretching $\lambda_1 = \lambda_2$. The deformation gradient is then of the form

$$\boldsymbol{F} = \lambda_1(\boldsymbol{1} - \vec{e}_3 \otimes \vec{e}_3) + \lambda_3\,\vec{e}_3 \otimes \vec{e}_3, \tag{5}$$

where $\{\lambda_i\}_{i=1,2,3}$ are the (principal) stretches along the respective directions $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$. At any material point, the Jacobian of the transformation is

$$J = \lambda_1^2\lambda_3. \tag{6}$$

3. Modeling with the $N = 1$-Ogden model

We start this paper by chosing for the part ψ_{iso} in (1) an $N = 1$-Ogden type model, Ogden (1997). In terms of the modified principal stretches $\bar{\lambda}_A = J^{-1/3}\lambda_A$, $A = 1, 2, 3$, we write

$$\psi_{\text{iso}} = \frac{2\mu}{m_1^2}\big(\bar{\lambda}_1^{m_1} + \bar{\lambda}_2^{m_1} + \bar{\lambda}_3^{m_1} - 3\big). \tag{7}$$

where μ denotes the classical shear modulus in the reference configuration, known from the linear theory, and m_1 is Ogden's coefficient.

Next, the principal isochoric stresses are given in terms of the principal values of the Kirchhoff-type isochoric stress tensor $J\boldsymbol{\sigma}_{\text{iso}}$ as

$$J\sigma_{\text{iso}_A} \equiv \lambda_A\frac{\partial\psi_{\text{iso}}}{\partial\lambda_A} = \bar{\lambda}_A\frac{\partial\psi_{\text{iso}}}{\partial\bar{\lambda}_A} - \frac{1}{3}\sum_{B=1}^{3}\Big(\bar{\lambda}_B\frac{\partial\psi_{\text{iso}}}{\partial\bar{\lambda}_B}\Big), \tag{8}$$

where use has been made of the following useful relations, see for example Simo and Taylor (1991) for details:

$$\frac{\partial J}{\partial\lambda_A} = J\lambda_A^{-1} \quad \text{and} \quad \frac{\partial\bar{\lambda}_B}{\partial\lambda_A} = J^{-\frac{1}{3}}\big(\delta_{AB} - \frac{1}{3}\bar{\lambda}_B\bar{\lambda}_A^{-1}\big).$$

Now back to our sample example, from (5), (6) and (8), we have

$$\begin{aligned} J\sigma_{\text{iso}_1} = J\sigma_{\text{iso}_2} &= \frac{2\mu}{3m_1}\left[\Big(\frac{\lambda_1}{\lambda_3}\Big)^{\frac{m_1}{3}} - \Big(\frac{\lambda_3}{\lambda_1}\Big)^{\frac{2m_1}{3}}\right], \\ J\sigma_{\text{iso}_3} &= \frac{4\mu}{3m_1}\left[\Big(\frac{\lambda_3}{\lambda_1}\Big)^{\frac{2m_1}{3}} - \Big(\frac{\lambda_1}{\lambda_3}\Big)^{\frac{m_1}{3}}\right]. \end{aligned} \tag{9}$$

3.1. Volumetric strain-energy and apparent Poisson's ratio

After many attempts, we have decided to choose the following form for the volumetric part of the strain-energy function:

$$\psi_{\text{vol}}(J) = \frac{\kappa}{m_1}\Big(\frac{J^{\alpha+1} - 1}{\alpha + 1} - \frac{J^{1-\beta} - 1}{1 - \beta}\Big), \tag{10}$$

with the constant κ interpreted as the elastic bulk modulus in the reference configuration. The empirical dimensionless coefficients α and β must, *if possible*, be determined later on in such a way that the apparent zero Poisson's ratio property can be satisfied. This determination strongly depends on the model used for the isochoric part of the response, here the one given by (7). Finally, the m_1 dimensionless constant in (10) is the one already used in (7) as well. The volumetric part of the stress tensor is then, see Eq. (2),

$$\boldsymbol{\sigma}_{\text{vol}} = \frac{\kappa}{m_1}(J^{\alpha} - J^{-\beta})\,\boldsymbol{1}. \tag{11}$$

It immediately follows from (11) that, as κ is the bulk modulus in the reference configuration, the connection

$$m_1 = \alpha + \beta, \tag{12}$$

must be satisfied. Indeed, at the limiting case of an infinitesimal perturbation near the reference configuration, the first order development of the expression (11) must be identical to the volumetric part of the constitutive relation in the infinitesimal theory.

Back again to our example, replacing the Jacobian (6) into (11) with the use of (12), gives the volumetric part of the stress as

$$\boldsymbol{\sigma}_{\text{vol}} = \frac{\kappa}{m_1}\big(\lambda_1^{2\alpha}\lambda_3^{\alpha} - \lambda_1^{2(\alpha-m_1)}\lambda_3^{\alpha-m_1}\big)\,\boldsymbol{1}, \tag{13}$$

that completes the expression of the stress tensor (2) for our case. In particular, along the directions $\vec{e}_1$ and $\vec{e}_2$, the identical transverse stress components are zero, see Eq. (4), and hence $J\sigma_{11} = J\sigma_{22} = 0$. We then get from $(9)_1$ and (13):

$$\begin{aligned} &\frac{\kappa}{m_1}\big(\lambda_1^{2(\alpha+1)}\lambda_3^{\alpha+1} - \lambda_1^{2(\alpha-m_1+1)}\lambda_3^{\alpha-m_1+1}\big) \\ &\quad + \frac{2\mu}{3m_1}\big(\lambda_1^{\frac{m_1}{3}}\lambda_3^{-\frac{m_1}{3}} - \lambda_1^{-\frac{2m_1}{3}}\lambda_3^{\frac{2m_1}{3}}\big) = 0. \end{aligned} \tag{14}$$

Now if the apparent Poisson's ratio is zero in the whole strain range, then the lateral stretches $\lambda_1 = \lambda_2$ must always be equal to 1 for any admissible stretch λ_3 along the uniaxial loading. From (14), by imposing $\lambda_1 = 1$ we get

$$\frac{\kappa}{m_1}\big(\lambda_3^{\alpha+1} - \lambda_3^{\alpha-m_1+1}\big) + \frac{2\mu}{3m_1}\big(\lambda_3^{-\frac{m_1}{3}} - \lambda_3^{\frac{2m_1}{3}}\big) = 0, \tag{15}$$