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A note on the higher order strain and stress tensors within deformation gradient elasticity theories: Physical interpretations and comparisons

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ABSTRACT

Higher order strain and stress tensors encompassed within gradient elasticity theories are discussed with a particular concern to the physical meaning of double and triple stresses. A single rule is shown to hold for the physical interpretation of the indices of a higher order stress tensor both within distortion gradient and strain gradient theories, whereas the analogous Mindlin's rule holds only within distortion gradient theories. Double and triple stresses are discussed separately with the aid of simple illustrative examples. A *corrigendum* to a previous paper by the author (IJSS 50 (2013) 3749–3765) is also presented.

1. Premise

Gradient theories of solid mechanics (elasticity, plasticity, damage mechanics and the like) are nowadays accepted by the research community as effective analytical tools to address a large variety of structural problems in which the effects of the micro-structural inhomogeneities cannot be disregarded. As it emerges from the related extensive literature (see e.g. the review papers by Askes and Aifantis, 2011; Javili et al., 2013; and the book by Gurtin et al., 2010), these gradient theories involve complex concepts of strain and stress states which need the use of tensors of order higher than two, to which researchers are likely not well acquainted.

In particular within elasticity, two different types of deformation gradient theories have emerged, of which one is featured by a strain energy function depending on the strain, ε_{ij}, along with the gradient(s) of the (compatible) distortion, $h_{ij} := \partial_i u_j$, (Mindlin, 1965; Mindlin and Eshel, 1968; Wu, 1992); the other is featured by a strain energy function depending on the strain, ε_{ij}, and on its gradient(s) (dell'Isola et al., 2009; Exadaktylos and Vardulakis, 2001; Lazar et al., 2006; Polizzotto, 2013). These gradient elasticity theories were distinguished by Mindlin and Eshel (1968) as Form I and Form II theories, respectively, but more frequently they are both referred to as "strain gradient elasticity" theories within the literature. For more clarity, the above theories are here distinguished with the names of *distortion gradient elasticity* (DGE) theory the former, *strain gradient elasticity* (SGE) theory the latter.[1] These theories lead to boundary-value problems featured by a same displacement partial differential equation system (Mindlin, 1965; Mindlin and Eshel, 1968), however the inherent notions of double and triple stresses exhibit conceptual and qualitative differences which are in general not sufficiently clarified. Recently, the symmetry features associated to the mentioned Form I and Form II were addressed by Gusev and Lurie (2015) in a variational formulation for a simplified isotropic material model of gradient elasticity endowed with only four constants (including the Lamé ones).

Explanations on the subject of higher order tensors as representations of double (or dipole) and triple (or quadrupole) stresses are available in the literature (see e.g. Gronwald and Held, 1993; Jaunzemis, 1967; Lazar and Maugin, 2005; Love, 1927; Mindlin, 1964). More recently, the present author addressed the subject in question with a particular concern to the extra indices of the higher order stress tensors, which represent the lever arm(s) of the double and triple stresses, see Appendix B in Polizzotto (2013). The purpose of the present note is to provide a further contribution aimed at improving the physical interpretation of the higher order strain and stress tensors, making a clear distinction between DGE and SGE theories. A second gradient theory will be considered, which will give us the opportunity to revise the content of Appendix B in Polizzotto (2013). The question of which model

[1] In the literature, the wording "second displacement gradient" is often used in place of the nonstandard "(first) distortion gradient". The two parallel concepts of "distortion" and "strain" are here believed to be more suitable to distinguish the above twin theories from each other, since these theories ultimately are each a particular form of second displacement gradient theory.

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among DGE and SGE may be better than the other is not discussed in the present work. For presentation clarity, a non-standard self-explaining nomenclature is introduced.

2. Introductory concepts and basic analytical relations

In the following, a continuum referred to Cartesian orthogonal axes, say x_i, $(i = 1, 2, 3)$, is considered along with the standard index notation.

2.1. Distortion gradient elasticity (DGE) theory

The DGE theory is centered on a (Helmholtz) strain energy function as, say, $\psi_d = \psi_d(\varepsilon_{ij}, \partial_i h_{jk}, \partial_i\partial_j h_{kl})$, where $h_{ij} := \partial_i u_j$ is the (compatible) distortion tensor and $\varepsilon_{ij} = h_{(ij)} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the standard strain tensor. The tensor variables ε_{ij}, $\eta^{(1)}_{ijk} := \partial_i h_{jk}$ and $\eta^{(2)}_{ijkl} := \partial_i\partial_j h_{kl}$ describe basic deformation states of the generic volume element, namely: ε_{ij}, a uniform strain within the whole volume element; $\eta^{(1)}_{ijk}$, a distortion h_{jk} linearly varying in the x_i direction; $\eta^{(2)}_{ijkl}$, a distortion h_{kl} varying bi-linearly in the x_i and x_j directions.[2]

The distortion gradient variables $\eta^{(1)}_{ijk}$ and $\eta^{(2)}_{ijkl}$ possess the following symmetry features: $\eta^{(1)}_{ijk} = \partial_i\partial_j u_k$ is symmetric with respect to the first index pair (*i, j*), hence it contains $3\times 6 = 18$ independent components, whereas $\eta^{(2)}_{ijkl} = \partial_i\partial_j\partial_k u_l$ is symmetric with respect to the (first) index triple (*i, j, k*), hence it has $3\times 10 = 30$ independent components[3] (Mindlin, 1965). Both $\eta^{(1)}_{ijk}$ and $\eta^{(2)}_{ijkl}$ are *not* symmetric with respect to their own last index pair.

The stress state corresponding to a given deformation state is determined by the partial derivatives of ψ_d, that is,

$$\sigma_{ij} = \frac{\partial\psi_d}{\partial\varepsilon_{ij}}, \quad \tau^{(1)}_{ijk} = \frac{\partial\psi_d}{\partial\eta^{(1)}_{ijk}}, \quad \tau^{(2)}_{ijkl} = \frac{\partial\psi_d}{\partial\eta^{(2)}_{ijkl}} \tag{1}$$

which are the constitutive equations for the standard stresses σ_{ij}, the double stresses $\tau^{(1)}_{ijk}$,[4] and the triple stresses $\tau^{(2)}_{ijkl}$, all of which possess symmetry features like the corresponding work-conjugate strain-like variables. The stress power $W_d = \dot{\psi}_d$ proves to be expressed as

$$W_d = \sigma_{ij}\,\dot{\varepsilon}_{ij} + \tau^{(1)}_{ijk}\,\dot{\eta}^{(1)}_{ijk} + \tau^{(2)}_{ijkl}\,\dot{\eta}^{(2)}_{ijkl} \tag{2}$$

The stresses (1), together with the so-called *total stress* defined as

$$S_{ij} := \sigma_{ij} - \partial_p\tau^{(1)}_{pij} + \partial_p\partial_q\tau^{(2)}_{pqij} \tag{3}$$

are all required to satisfy the field and boundary equilibrium equations, for which we refer to Mindlin (1965) and Mindlin and Eshel (1968). Here we only observe that, since $\tau^{(1)}_{ijk}$ and $\tau^{(2)}_{ijkl}$, like the work-conjugate variables $\eta^{(1)}_{ijk}$ and $\eta^{(2)}_{ijkl}$, are not symmetric with respect to their own last index pair, then the total stress S_{ij} in (3) is *not symmetric*.

For possible choices of the function ψ_d in terms of invariants of the tensors ε_{ij}, $\eta^{(1)}_{ijk}$, $\eta^{(2)}_{ijkl}$ we refer to Mindlin (1965) and Mindlin and Eshel (1968).

2.2. Strain gradient elasticity (SGE) theory

The SGE theory is centered on a (Helmholtz) strain energy function as $\psi_s = \psi_s(\varepsilon_{ij}, \partial_i\varepsilon_{jk}, \partial_i\partial_j\varepsilon_{kl})$. The basic deformation states of the material element are described by ε_{ij} (uniform strain within the element), $\varepsilon^{(1)}_{ijk} := \partial_i\varepsilon_{jk}$ (strain ε_{jk} linearly varying in the x_i direction), and $\varepsilon^{(2)}_{ijkl} := \partial_i\partial_j\varepsilon_{jk}$ (strain ε_{jk} bi-linearly varying in the x_i and x_j directions).

The strain gradient variables $\varepsilon^{(1)}_{ijk}$ and $\varepsilon^{(2)}_{ijkl}$ are symmetric with respect to their own last index pair, the second one also with respect to the first index pair (*i, j*), therefore they possess, respectively, $3 \times 6 = 18$ and $6 \times 6 = 36$ independent components [5] (Lazar et al., 2006). The tensor $\varepsilon^{(1)}_{ijk}$ is *not* invariant with respect to interchanges of the first index *i* with anyone within the last index pair (except, obviously, in the case $i = j$ and/or $i = k$); the same can be stated for $\varepsilon^{(2)}_{ijkl}$, in the sense that none of the first two indices can be interchanged with anyone within the last index pair.

The stress state corresponding to a given set of strain-like variables is obtained as

$$\sigma_{ij} = \frac{\partial\psi_s}{\partial\varepsilon_{ij}}, \quad \sigma^{(1)}_{ijk} = \frac{\partial\psi_s}{\partial\varepsilon^{(1)}_{ijk}}, \quad \sigma^{(2)}_{ijkl} = \frac{\partial\psi_s}{\partial\varepsilon^{(2)}_{ijkl}} \tag{4}$$

which are the constitutive equations for the standard stress σ_{ij}, the double stress $\sigma^{(1)}_{ijk}$, and the triple stress $\sigma^{(2)}_{ijkl}$, all of which possess the same symmetry features as the related work-conjugate strain-like variables. Indeed, the same nomenclature is used in the literature for both stress sets (1) and (4). The stress power $W_s = \dot{\psi}_s$ proves to be expressed as

$$W_s = \sigma_{ij}\,\dot{\varepsilon}_{ij} + \sigma^{(1)}_{ijk}\,\dot{\varepsilon}^{(1)}_{ijk} + \sigma^{(2)}_{ijkl}\,\dot{\varepsilon}^{(2)}_{ijkl} \tag{5}$$

The stresses (4) together with the related *total stress* defined as

$$T_{ij} := \sigma_{ij} - \partial_p\sigma^{(1)}_{pij} + \partial_p\partial_q\sigma^{(2)}_{pqij} \tag{6}$$

must satisfy the field and boundary equilibrium equations, for which we make reference to Polizzotto (2013). We note that, since in (6) the variables $\sigma^{(1)}_{ijk}$ and $\sigma^{(2)}_{ijkl}$ are symmetric with respect to their own last index pair, then the total stress T_{ij} of (6) is *symmetric*. The latter property makes the strain-gradient based approach to gradient elasticity preferable to the analogous distortion-gradient based one.

A simple example of Helmholtz strain energy function for SGE can be expressed in the form

$$\psi_s = \frac{1}{2}C_{ijkl}\left[\varepsilon_{ij}\,\varepsilon_{kl} + (\ell_1)^2\partial_p\varepsilon_{ij}\,\partial_p\varepsilon_{kl} + (\ell_2)^4\partial_p\partial_q\varepsilon_{ij}\,\partial_p\partial_q\varepsilon_{kl}\right] \tag{7}$$

where ℓ_1 and ℓ_2 are internal length scale parameters and C_{ijkl} is the usual moduli tensor of isotropic elasticity, that is, denoting by λ, μ the Lamé constants,

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{8}$$

[2] For an $(m+n)$th order tensor, **A**, equal to the mth order gradient of a nth order tensor **B**, a rule often adopted in the literature (e.g. Exadaktylos and Vardulakis, 2001; Fleck and Hutchinson, 1997; Mindlin and Eshel, 1968) is followed here, whereby the first $m \geq 1$ indices denote the inherent gradient co-ordinates, that is, $A_{i_1\ldots i_m j_1\ldots j_n} = \partial_{i_1}\ldots\partial_{i_m}B_{j_1\ldots j_n}$, but many researchers (e.g. dell'Isola et al., 2009; Lazar et al., 2006) prefer to locate the mentioned indices in the last positions within the index string of **A**, such as to read $A_{j_1\ldots j_n i_1\ldots i_m} = \partial_{i_1}\ldots\partial_{i_m}B_{j_1\ldots j_n}$.

[3] The number of independent components of $\eta^{(2)}_{ijkl}$ is equal to the product of the numbers of the analogous components of a fully symmetric third order tensor (10) and of a vector (3). The independent components $\eta^{(2)}_{ijkl}$ can be recognized to be of the type: $3\times 3 = 9$ components of the type (*iiil*), $6\times 3 = 18$ of the type (*ijjl*), $1\times 3 = 3$ of the type (*ijkl*), in which $i \neq j \neq k$, (no sum on repeated indices).

[4] A symbol with a superscript $^{(1)}$ to denote a "double" stress may appear inappropriate. The superscript $^{(1)}$ on such a symbol just means that the double stress is associated to a "first" strain-like variable and possesses "one" lever arm. The same holds for the superscript $^{(2)}$ used for a triple stress, the latter being related to a "second" strain-like gradient and endowed with "two" lever arms.

[5] The tensor $\varepsilon^{(2)}_{ijkl} = \partial_i\partial_j\varepsilon_{kl}$, with ε_{kl} being an arbitrary (symmetric) strain tensor, has $6\times 6 = 36$ independent components, as stated above. If instead ε_{kl} is compatible, i.e. $\varepsilon_{kl} = u_{(k,l)}$, then the tensor $\varepsilon^{(2)}_{ijkl} = \frac{1}{2}(\partial_i\partial_j\partial_k u_l + \partial_i\partial_j\partial_l u_k)$ has $(30+30)/2 = 30$ independent components, just like $\eta^{(2)}_{ijkl}$. Indeed, the compatibility conditions make $\varepsilon^{(2)}_{ijkl}$ have the same number (30) of independent components as $\eta^{(2)}_{ijkl}$.