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A loaded Timoshenko beam bonded to an elastic half plane

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ABSTRACT

The problem of a Timoshenko beam of finite length loaded by concentrated forces and couples and perfectly bonded to a homogeneous elastic and isotropic half plane is considered in the present work. In particular, the effects induced by shear deformation of the beam on the contact stresses arising at the interface between the beam and the underlying half plane are investigated accurately. An asymptotic analysis of the stress field at the beam ends and in the neighborhood of the loaded section of the beam allows us to characterize the singular nature of the peeling and shear stresses. The problem is formulated by imposing the strain compatibility condition between the beam and the half plane, thus leading to a system of two singular integral equations with Cauchy kernel. The unknown interfacial stresses are expanded in series of Jacobi orthogonal polynomials displaying complex singularity. This approach allows us to handle the oscillatory singularity and to reduce the integral equations to a linear algebraic system of infinite equations for the unknown coefficients of the interfacial stresses, which is solved through a method of collocation. The interfacial peeling and shear stresses and, in turn, the displacement field along the contact region have been calculated under various loading conditions applied to the beam. The internal forces and bending moments along the beam have been calculated varying the shear and flexural stiffness of the beam. The complex stress intensity factors and the strength of the stress singularities have been assessed in detail.

1. Introduction

Contact problems between beams, plates, bars, strips etc. and an elastic substrate attracted a lot of interest in the field of solid mechanics in order to predict the mechanical behavior of a variety of composites systems, specially used in civil and mechanical engineering. As an example, steel panels and plates are often stiffened by metallic strip-like elements in order to increase their out-of-plane strength, their bending stiffness and, in turn, their buckling load. This is a key issue concerning offshore platforms, bridge decks and other kinds of structures which require high-performance mechanical behavior, avoiding excessive weight and materials consumption (e.g. Grondin et al., 1999).

Fibre reinforced polymer (FRP) bonding is widely used as an effective method to strengthen existing reinforced concrete (RC) elements, thus improving their resistance and toughness and, in turn, their service life. Typical failure mode affecting this kind of systems occurs with an interfacial crack between the FRP stiffener and the RC plate, growing toward the free ends of the stiffener (Oehlers and Seracino, 2004).

In the framework of civil engineering, simplified analyses of the soil-structure interaction are usually performed by modeling the building foundations like beams (for strip-like foundation) or plates (for raft foundation) supported by an elastic half plane (e.g. Wolf, 1988).

In the last decades, the scientific community has focused its attention in renewable energy generators. Among these, a promising technique to produce “green” energy consists in applying piezoelectric patch-like or strip-like transducers to existing flexible structures (typically, cantilever beams and walls), thus converting the vibration motion of the hosting structural elements into electrical energy. Similarly, smart sensors and actuators can be applied to existing structures to monitoring their mechanical behavior in time (e.g. Lin and Liu, 2006).

In microelectronics, many MEMS and NEMS like bulk acoustic resonators, high density capacitors, coplanar plate varactors, skin-like circuits, crystalline undulators (e.g. Guidi et al., 2007; Lanzoni et al., 2008; Lanzoni and Radi, 2009) and other miniaturized packages involve thin films and coatings deposited onto a substrate (Gevorgian, 2009; Shen, 2010). These microsystems are often

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subjected to high residual stresses taking place during their fabrication process (a comprehensive reference about the mechanical behavior of piezoelectric multilayer actuators can be found in Ballas, 2007).

In order to predict the mechanical response of this kind of composite structures, a proper investigation of the stress and strain fields arising at the interface is mandatory. Special attention must be paid to stress and strain concentrations which can lead to loss of adhesion, debonding and other damaging phenomena affecting the durability and stability of this kind of devices.

In many studies, films (or stiffeners) bonded to a compliant substrate are modeled like bars or membranes, neglecting their bending stiffness. Such an assumption allows ignoring the interfacial peeling stress. As an example, Arutiunian (1968) solved the problem of a thin film bonded to an elastic half plane subjected to a thermal variation. Later, a simpler method was adopted by Erdogan and Gupta (1971) and Morar and Popov (1971), who solved the problem of a strip bonded to a half plane subjected to axial loads applied at the ends of the coating. Using a similar approach, Guler (2008) considered the problem of thin cover plates bonded to a graded substrate. A detailed analysis about the stress singularity in thin films welded to an elastic half plane under various loading conditions can be found in Lanzoni (2011) (for the stress singularities of a membrane stiffener with variable thickness see also Erdogan and Ozturk, 2008). A bar model has been adopted by Villaggio (2003) to study the brittle detachment of a stiffener welded to an elastic plate.

A number of numerical analyses have been also performed to study the mechanical behavior of single or multi-layered systems involving thin films and coatings. As an example, finite element–boundary integral equation methods (FE-BIE) have been widely used to investigate bars and membranes bonded to an elastic half plane (e.g. Takahashi and Shibuya, 1997, 2003; Tullini et al, 2012). In these studies, the problem is devised by using a mixed variational formulation involving the Green function for the half plane.

Nonetheless, if the coating flexural stiffness becomes significant, then the membrane models become inappropriate and the bending rigidity of the coating must be necessarily taken into account. In this case, both shear and peeling stresses arise at the interface between the cover and the substrate.

One of the earliest studies concerning the contact problem among beam elements has been performed by Timoshenko (1925). This author studied a bimetal strip under bending or thermal loads and solved the problem by imposing the compatibility between the axial strains of both beams in contact, neglecting the occurrence of interfacial stresses. Later, the approach has been extended by Suhir (1986) by introducing "interfacial compliance" parameters to describe the deformation of the cross section of the beams under the shearing load over the beam thickness. Moreover, the interfacial shear and peeling stresses are found as exponential functions of the axial coordinate. This method has been employed to investigate the effects induced by thermal stress in multilayered structures (Suhir, 1988) and the peeling stress as a function of a "through-thickness" spring parameter (Suhir, 1989). Indeed, the approach adopted by Timoshenko (1925) gives exact results only if the Young modulus multiplied by the square of the thickness for the two strips is the same (Moore and Jarvis, 2004).

Shield and Kim (1992) performed an analytical study dealing with an Euler-Bernoulli beam welded to an elastic half plane under symmetric loads applied at the ends of the beam. These Authors expanded the interfacial stresses in series of orthogonal Chebyshev polynomials displaying square root singularity at the beam ends. As pointed out by the Authors, the membrane approximation may provide rough predictions, in particular for systems sensitive to mode I failure.

However, the Euler–Bernoulli beam model cannot be adopted for analyzing elements characterized by a significant shear deformation, like short beams, or when the constituent material is compliant with respect to shear loads, like for FRP profiles, whose polymeric (thermoset or thermoplastic) matrix exhibits low shear strengths and shear moduli (Barbero, 1999). In such cases, the Timoshenko beam model reveals more effective than the Euler–Bernoulli theory, in particular for dynamical analyses. As an example, transversal vibrations of railways have been studied taking into account their cross-sectional deformation by modeling the railway track as Timoshenko beams on elastic ground (Wu et al., 1999). Moreover, Timoshenko beam theory has been recently used to investigate the vibration frequencies of carbon nanotubes characterized by small length-to-diameter ratios (e.g. Wang et al., 2006).

Only few analytical studies dealing with Timoshenko beams in contact with elastic substrates can be found in the Literature. Among these, Essenburg (1962) considered a Timoshenko beam supported by a Winkler foundation and he found a closed form solution of the governing equation. Li et al. (1988) studied the unilateral frictionless contact between a Timoshenko beam resting on an elastic layer supported by a rigid base by using a Gauss–Chebyshev quadrature method. Bjarnehed (1993) performed a numerical investigation of the problem of a Timoshenko beam resting on an elastic cushion bonded to an orthotropic half plane. Tezzon et al. (2015) used a coupled FE-BIE method to model shear deformable beams bonded to an isotropic elastic half-space. However, to the authors knowledge, a closed form solution of the interfacial stress field of a Timoshenko beam bounded to a half plane under various loading conditions has not been found yet.

In the present work, the contact problem of a Timoshenko beam of finite length bonded to a homogeneous elastic half plane is investigated. The proposed approach consists in the imposition of the strain compatibility condition between the beam and the half plane by expanding the interfacial stresses in series of orthogonal Jacobi polynomials. This representation allows to remove the singularity of the integral equations derived from the strain compatibility condition. Then, by using a collocation technique, the problem is reduced to a linear algebraic system of equations for the unknown coefficients of the series expansions for the interfacial stresses. The present study allows to investigate the distribution of peeling and shear interfacial stresses within the contact region varying the stiffness parameters of the beam, with particular emphasis to the effects of shear deformation. A detailed analysis of the strength of the stress singularities at the ends of the beam is provided. In this respect, the present study represents an extension of the work performed by Shield and Kim (1992).

The paper is organized as follows. In Section 2, the governing equations for the beam and the half plane are reported and the method used to solve the singular integral equations is discussed therein. An asymptotic analysis of the interfacial stresses at the beam ends is performed in Section 3.1. The singular nature of the interfacial stresses in the neighborhood of a concentrated load applied at the inner of the beam is investigated in Section 3.2. Some relevant loading conditions are discussed in detail in Section 4. The main results in terms of interfacial stresses distribution, stress intensity factors and strength of stress singularities are reported for some typical symmetric as well as skew-symmetric loading conditions in Section 5. Finally, conclusions are drawn in Section 6.

2. Governing equations

Let us consider a Timoshenko beam of length $2a$ with a rectangular cross section of height h and unitary width, subjected to a system of axial and shear forces (N_1, N_2), (T_1, T_2) and bending moments (M_1, M_2) applied at both ends of the beam (shear forces are taken positive if upward directed; axial forces are