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Variational formulations, instabilities and critical loadings of space curved beams

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ABSTRACT

Beam theories have been extensively studied for applications in structural engineering. Space curved beams with large displacements, however, have been explored to a much less extent, not to mention explicit solutions concerning instabilities and critical loadings. In this paper, by carefully accounting for geometric nonlinearity and different scalings of kinematic variables, we present a variational framework for large-displacement space curved beams. We show that the variational formulation is consistent with the classic field equations, derive the appropriate boundary value problems for a variety of loading conditions and kinematic constraints, and generalize the Kirchhoff's helical solutions. Explicit planar solutions for semi-circular arches are obtained upon linearization. Further, two nonlinear asymptotic theories are proposed to address ribbon-like and moderately deformed curved beams, respectively. Based on the method of trial solutions, we obtain explicit approximate solutions to critical loadings for semi-circular arches losing stabilities due to twisting and out-of-plane displacement. The variational framework, nonlinear asymptotic theories, stability analysis and explicit solutions are anticipated to have novel applications in stretchable electronics and biological macromolecules.

1. Introduction

Upon making suitable kinematic hypotheses, there are two approaches to formulating an effective theory for lower dimensional elastic bodies. In the field-equation approach, the concept of forces/moments is primitive, the balance laws relating internal forces/moments and external forces/moments are derived by free-body-diagram analysis, and finally, constitutive laws relating internal forces/moments and kinematic variables are postulated to close the system. In the variational approach, the concept of free/strain energy is primitive, and upon postulating the functional dependence of strain energy on kinematic variables, the field equations follow as the Euler–Lagrange equations of the variational principles, e.g., the Hamilton's principle or the principle of minimum free energy. The two approaches shall always yield equivalent, though sometimes not obvious, boundary value problems for self-consistency if the kinematic and constitutive hypotheses are the same in these two approaches.

In this paper we formulate nonlinear variational theories for curved beams, which are motivated by novel applications in stretchable electronics and biological macromolecules. To achieve high electrical performance and mechanical reliability, stretchable electronics have to leverage intrinsically stiff but well established inorganic materials like metal and silicon. A reliable way to build continuous, stretchable structure out of stiff materials is the serpentine design, i.e., meandering ribbons or wires (Fig. 1). When stretched end-to-end, serpentine ribbons or wires can rotate in plane as well as buckle out of plane to accommodate the applied displacement, resulting in greatly reduced local elastic strains and much lower effective stiffness (Li et al., 2005; Su et al., 2012; Widlund et al., 2014; Zhang et al., 2014). These features enable applications ranging from tissue-like bio-integrated electrodes (Kim et al., 2011; Yeo et al., 2013), micro-heaters (Yu et al., 2013), deformable solar cells (Tang et al., 2014), transparent stretchable conductors (Yang et al., 2015), soft nanogenerators (Ma et al., 2013) to deployable sensor networks (Lanzara et al., 2010) and coronary stents (Mani et al., 2007). However, in spite of recent efforts in plane

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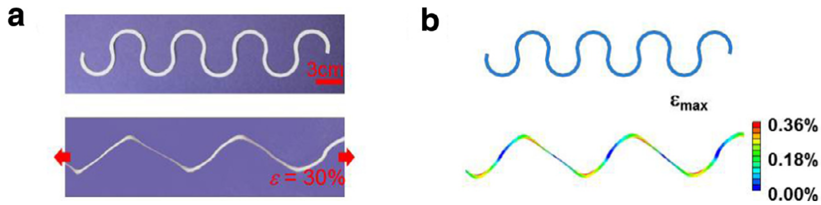

Fig. 1. A serpentine ribbon buckles out of plane when stretched end-to-end: (a) experimental observation of a paper ribbon with 30% end-to-end elongation, and (b) finite element model (FEM) results showing the maximum principle strain in the corresponding ribbon.

strain modeling of freestanding serpentines (Widlund et al., 2014), buckling analysis of thin freestanding serpentine ribbons (Zhang et al., 2013), and analytical and numerical modeling of self-similar serpentines (Su et al., 2015; Zhang et al., 2014), the designs of the serpentine shape are still largely empirical, particularly for serpentines of extreme cross-sectional aspect ratios and undergoing large out-of-plane deformations. A general, preferably variational, framework will be convenient for stability analysis and rational design of high-performance serpentines for stretchable electronics. Meanwhile, it has been a standard practice to model macromolecules such as DNA, polymers and proteins, as an elastic rod for their mechanical behaviors, see e.g. the textbook of Doi and Edwards (1989), review article of Manning (1985), series of works of Zhou et al. (1999, 2000, 2002), and references therein. Though it has been shown that the worm-like-chain (WLC) model, i.e., a uniform circular elastic rod under bending, predicts reasonable force-versus-extension relation of DNA strands beyond a few kilobase-pair range (Smith et al., 1996, 1992). At a lengthscale of tens of base pairs, a more precise description of DNA is necessary to account for the anisotropy, twisting and kinks of DNA structures (Hoffman, 2004; Noy and Golestanian, 2012; Wiggins et al., 2005). Moreover, depending on the salt concentration of the ambient solution, the natural (i.e., stress-free or ground) state of the DNA is not a straight chain, but admits a variety of supercoiling configurations (Manning, 1985). It is of great interest to include effects of charge screening and electrostatic interactions and to carry out statistical mechanics analysis for DNA. These purposes demand a variational framework, i.e., a Hamiltonian in terms of reasonable set of kinematic variables.

Though many of the essential components of a general 3D curved beam theory have been investigated more than 150 years ago in the works of Kirchhoff (Love, 1944), our variational framework accounting for the geometric nonlinearity of large displacements is simple, self-contained and ready for novel applications in stretchable electronics and biological macromolecules. We systematically derive general boundary conditions and find some inconsistency in earlier works. The variational formulation is particularly convenient for rigorous analysis by the direct method of calculus of variations and for investigating beams with extreme cross-sectional aspect ratios (i.e., ribbons). However, the fully nonlinear theory is not prone to explicit solution on one hand, on the other hand, the linearized theory cannot address instabilities due to twisting and out-of-plane displacement. Therefore, we propose some simplified nonlinear theories and explicitly calculate the critical loadings by the method of trial solutions. More accurate solutions on the critical loadings and stabilities of equilibrium states can be achieved by numerical methods.

For classical applications in structural engineering, there are many works on elastic theories of rods in the literature which are too voluminous to recount here. For historical references, the reader may consult Love's treatise (Love, 1944) and Antman's survey (Antman and Truesdell, 1973). As for space curved beams, Reissner (1973); 1981) pioneered a finite strain theory that was later refined by subsequent works of Simo (1985), Simo and Vu-Quoc (1986), and Iura and Atluri (1988, 1989). The numerical aspect of space-curved beam models has been a particularly active research area in the last thirty years with contributions from, e.g., Petrov and Geradin (1998), Ishaqduddin et al. (2012), Saje et al. (2012) and references therein. Alternatively, the theory of an elastic rod can be reformulated as a one-dimensional Cosserat or micropolar theory (Cosserat and Cosserat, 1909); kinematic relations and balance laws can be conveniently explored using Clifford or geometric algebra (McRobie and Lasenby, 1999). In this model, each material point admits rotational degrees of freedom represented by a triad of orthonormal vectors in additional to the usual translational degrees of freedom. Some of the fundamental questions such as the existence, uniqueness and stability of a solution may be more conveniently addressed in the Cosserat framework (James, 1981; Steigmann and Faulkner, 1993).

The paper is organized as follows. We begin with the kinematic hypotheses and calculate the strain energy in Section 2.1. We formulate the variational principle and derive the associated Euler–Lagrange equations and boundary conditions in Section 2.2, and find that Kirchhoff's helical solutions can be applied to more general boundary conditions in Section 2.3. In Section 2.4 and 2.5, the geometrically nonlinear theory is linearized and solved for semi-circular arches with clamped supports, simple supports and cantilever. We propose two simplified nonlinear theories in Section 3.1, and obtain explicit solutions to critical loadings in Section 3.2. We conclude and summarize in Section 4. In the Appendix, we show our variational formulation is consistent with the existing field-equation approach.

Notation. We employ direct notation for brevity if possible. Vectors are denoted by bold symbols such as **e**, **u**, etc. When index notations are in use, the convention of summation over repeated index is followed. The inner (or dot) product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is defined as $\mathbf{a} \cdot \mathbf{b} := (\mathbf{a})_i (\mathbf{b})_i$, and the cross product $(\mathbf{a} \times \mathbf{b})_i := \mathcal{E}_{ijk} (\mathbf{a})_j (\mathbf{b})_k$, where $\mathcal{E}_{ijk}$ is the Levi-Civita symbol.

2. A variational formulation for space curved beams

2.1. Kinematics and strain energy

Consider a curved beam in space as illustrated in Fig. 2. In the reference configuration (Fig. 2(a)), the centroid line of the beam is a space curve with arc-length parameterization given by $\{\mathbf{c}_0(\xi^1) : 0 \le \xi^1 \le L\} \subset \mathbb{R}^3$. For simplicity, we assume the centroid curve remains to be of C^3-class (continuously differentiable up to the third order) with nonzero curvature in Sections 2-3 and postpone our discussion about less regular curves to Section 4. Let

$$\tilde{\mathbf{e}}_1(\xi^1) = \mathbf{c}_0{}'(\xi^1), \quad \tilde{\mathbf{e}}_2(\xi^1) = \frac{\mathbf{c}_0{}''(\xi^1)}{|\mathbf{c}_0{}''(\xi^1)|},$$

$$\tilde{\mathbf{e}}_3(\xi^1) = \tilde{\mathbf{e}}_1(\xi^1) \times \tilde{\mathbf{e}}_2(\xi^1)$$

be the local orthogonal Frenet frame ($' = d/d\xi^1$), and

$$\kappa_0(\xi^1) = \tilde{\mathbf{e}}_2(\xi^1) \cdot \tilde{\mathbf{e}}_1'(\xi^1) \qquad (\text{resp. } \tau_0(\xi^1) = \tilde{\mathbf{e}}_2' \cdot \tilde{\mathbf{e}}_3(\xi^1))$$

be the curvature (resp. torsion) of the space curve. Denote by $\mathcal{A}_0(\xi^1)$ the cross-sectional area normal to $\tilde{\mathbf{e}}_1(\xi^1)$ and $\mathcal{B}_0$ the reference, stress-free and undeformed elastic body of the beam. In the