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# A methodology for the estimation of the effective yield function of isotropic composites

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ABSTRACT

In this work we derive a general model for $N-$phase isotropic, incompressible, rate-independent elasto-plastic materials at finite strains. The model is based on the nonlinear homogenization variational (or modified secant) method which makes use of a linear comparison composite (LCC) material to estimate the effective flow stress of the nonlinear composite material. The homogenization approach leads to an optimization problem which needs to be solved numerically for the general case of a $N-$phase composite. In the special case of a two-phase composite an analytical result is obtained for the effective flow stress of the elasto-plastic composite material. Next, the model is validated by periodic three-dimensional unit cell calculations comprising a large number of spherical inclusions (of various sizes and of two different types) distributed randomly in a matrix phase. We find that the use of the lower Hashin–Shtrikman bound for the LCC gives the best predictions by comparison with the unit cell calculations for both the macroscopic stress-strain response as well as for the average strains in each of the phases. The formulation is subsequently extended to include hardening of the different phases. Interestingly, the model is found to be in excellent agreement even in the case where each of the phases follows a rather different hardening response.



## 1. Introduction

The present work deals with the analytical and numerical estimation of the effective as well as the phase average response of $N-$phase incompressible isotropic elasto-plastic metallic composites. Special attention is given to particulate microstructures, i.e., composite materials which can be considered to comprise a distinct matrix phase and an isotropic distribution of spherical particles (Willis et al., 1982) (or in a more general setting an isotropic distribution of phases (Willis, 1977)). In the present study, the particles are considered to be stiffer than the matrix phase, which is the case in most metallic materials of interest, such as TRIP steels, dual phase steels, aluminum alloys and others. Such materials, usually contain second-phase particles (e.g., intermetallics, carbon particles) or just second and third phase variants (e.g., retained austenite, bainite, martensitic phases). In addition, these phases/particles tend to reinforce the yield strength of the composite while they usually have different strength and hardening behavior than the host matrix phase.

In the literature of nonlinear homogenization there exists a large number of studies for two-phase composite materials. The reader is referred to Ponte Castañeda and Suquet (1998), Ponte Castañeda (2002), Idiart et al. (2006), and Idiart (2008) for a review of the nonlinear homogenization schemes such as the ones used in the present work and relevant estimates. Nonetheless, very few studies exist in the context of three- or $N-$phase rate independent elasto plastic composites.

In view of this, the present work uses the nonlinear variational homogenization method (Ponte Castañeda, 1991) or equivalently the modified secant method (Suquet, 1995), which makes use of a linear comparison composite (LCC) material, to estimate the effective response of a $N-$phase nonlinear composite material. Even though, this method exists for several years most of the studies in the context of composite materials have been focused on two-phase composites where the optimization process required by the method can be done analytically (see for instance deBotton and Ponte Castañeda (1993)). Nevertheless, as the number of phases

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increases to three or more the optimization can only be done numerically. Perhaps, that is the reason that in his original work, Ponte Castañeda (1992) proposed general expressions (and bounds) for $N-$phase composites, but its numerical/analytical resolution remained untractable until today due to the complex optimization procedures required by the nonlinear homogenization method.

It should be pointed out at this point that these homogenization theories treat separately the elastic (which in the present case is trivial) and the plastic homogenization problem. That of course has certain impact if cyclic loading is considered which is beyond the scope of the present work and is not considered here. Nevertheless, recently, Lahellec and Suquet (2007) proposed an incremental variational formulation for materials with a hereditary behavior described by two potentials: a free energy and a dissipation function. This method has been introduced mainly to deal with the coupled elasto-plastic response of composites in an attempt to resolve the cyclic response of these materials (see also recent work by Brassart et al. (2011)). Note that these more advanced methods use the aforementioned or variants of the LCC estimates. In this regard, the present study, albeit not using this coupled scheme, reveals the nature of equations required to deal with a general $N-$phase composite material and could be potentially useful in the future for such more complete incremental schemes, which are based upon those simpler LCC homogenization theories.

### 1.1. Scope of the present work and major results

The scope of the present work is to provide a semi-analytical model for $N-$phase isotropic, incompressible rate-independent elasto-plastic materials. Simple analytical expressions are given for the effective yield stress of a two-phase composite (see also (deBotton and Ponte Castañeda, 1993)), while a simple semi-analytical expression (requiring the solution of a constrained optimization problem for $N-1$ scalar quantities) is given for the $N-$phase composite. Additional analytical expressions are also provided for the phase concentration tensors and average strains in each phase in terms of the aforementioned optimized scalar quantities. In the context of two- and three-phase materials the model is assessed by appropriate three-dimensional multi-particle two- and three-phase periodic unit cell calculations considering both hardening and non-hardening phases. The agreement is found to be good not only for the effective yield stress but also for the phase average strains thus allowing for the extension of this model to include arbitrary isotropic hardening of the phases.

Specifically, we use the methodology developed by Ponte Castañeda and co-workers (Ponte Castañeda, 1991; Suquet, 1995) to derive a model for the rate-independent elastoplastic behavior of a macroscopically isotropic composite comprising $N$ phases. When the constituent phases are perfectly plastic the corresponding flow stress of the composite material $\tilde{\sigma}_0$ is determined from the solution of a constrained optimization problem:

$$\tilde{\sigma}_0=\sqrt{\inf_{\substack{y^{(i)}\geq 0\\ y^{(1)}=1\\ i=2,\dots,N}}\left(\sum_{r=1}^{N}c^{(r)}{\sigma_0^{(r)}}^2y^{(r)}\right)\left(\sum_{p=1}^{N}\frac{c^{(p)}}{3\,y^{(p)}+2\,y_0}\right)\left(\sum_{s=1}^{N}\frac{c^{(s)}\,y^{(s)}}{3\,y^{(s)}+2\,y_0}\right)^{-1}}.\tag{1}$$

where $N$ is the number of phases, $(c^{(i)},\sigma_0^{(i)})$ are the volume fraction and flow stress of phase $i$, and $y^{(i)}$ are positive optimization parameters. In turn, $y_0$ is a reference scalar to be chosen according to various linear homogenization schemes. For instance, best results are obtained with the well known Hashin–Shtrikman lower bound choice, i.e., $y_0=y^{(1)}=1$.

In the special case of a two-phase composite $(N=2)$, the optimization problem is solved analytically and the estimate for the composite flow stress becomes

$$\frac{\tilde{\sigma}_0}{\sigma_0^{(1)}}=\begin{cases}\frac{5\,c^{(2)}\,r+c^{(1)}\sqrt{9+6\,c^{(2)}-6\,c^{(2)}\,r^2}}{3+2\,c^{(2)}} & \text{if}\quad 1\leq r\leq 5\Big/\sqrt{4+6\,c^{(2)}},\\ \frac{1}{2}\sqrt{4+6\,c^{(2)}} & \text{if}\quad r\geq 5\Big/\sqrt{4+6\,c^{(2)}},\end{cases}\tag{2}$$

where $r=\sigma_0^{(2)}/\sigma_0^{(1)}$ is the contrast ratio. The predictions of the homogenization model agree well with the predictions of detailed three-dimensional unit cell finite element calculations as shown in the following.

The homogenization technique provides also accurate estimates for the average strains in the constituent phases. These estimates form the basis for the development of an approximate analytical model for the elastoplastic behavior of a composite with hardening phases. A method for the numerical integration of the resulting elastic-plastic equations is developed and the model is implemented into the ABAQUS general purpose finite element code. The predictions of the model agree well with the results of detailed unit cell finite element calculations of a composite with hardening phases.

Standard notation is used throughout. Boldface symbols denote tensors the orders of which are indicated by the context. The usual summation convention is used for repeated Latin indices of tensor components with respect to a fixed Cartesian coordinate system with base vectors $\mathbf{e}_i\,(i=1,2,3)$. The prefice det indicates the determinant, a superscript $T$ indicates the transpose, and the subscripts $s$ and $a$ the symmetric and anti-symmetric parts of a second-order tensor. A superposed dot denotes the material time derivative. Let $\mathbf{A}$, $\mathbf{B}$ be second-order tensors, and $\mathbf{C}$, $\mathbf{D}$ fourth-order tensors; the following products are used in the text: $(\mathbf{A}\cdot\mathbf{B})_{ij}=A_{ik}\,B_{kj}$, $\mathbf{A}:\mathbf{B}=A_{ij}\,B_{ij}$, $(\mathbf{A}\mathbf{B})_{ijkl}=A_{ij}\,B_{kl}$, $(\mathbf{C}:\mathbf{A})_{ij}=C_{ijkl}\,A_{kl}$, and $(\mathbf{C}:\mathbf{D})_{ijkl}=C_{ijpq}\,D_{pqkl}$. The inverse $\mathbf{C}^{-1}$ of a fourth-order tensor $\mathbf{C}$ that has the "minor" symmetries $C_{ijkl}=C_{jikl}=C_{ijlk}$ is defined so that $\mathbf{C}:\mathbf{C}^{-1}=\mathbf{C}^{-1}:\mathbf{C}=\mathcal{I}$, where $\mathcal{I}$ is the symmetric fourth-order identity tensor with Cartesian components $\mathcal{I}_{ijkl}=(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk})/2$, $\delta_{ij}$ being the Kronecker delta.

## 2. Power-law creep and perfect plasticity

We consider an incompressible creeping solid characterized by a power-law stress potential $U$ of the form

$$U(\sigma_e)=\frac{\sigma_0\,\dot{\varepsilon}_0}{n+1}\left(\frac{\sigma_e}{\sigma_0}\right)^{n+1},\tag{3}$$

where $\sigma_0$ is a reference stress, $\dot{\varepsilon}_0$ a reference strain rate, $n$ the creep exponent $(1\leq n\leq\infty)$, $\sigma_e=\sqrt{\frac{3}{2}\,\mathbf{s}:\mathbf{s}}$ the von Mises equivalent stress, $\boldsymbol{\sigma}$ the stress tensor, $p=\sigma_{kk}/3$ the hydrostatic stress, and $\mathbf{s}=\boldsymbol{\sigma}-p\,\boldsymbol{\delta}$ the stress deviator, $\boldsymbol{\delta}$ being the second-order identity tensor. The corresponding deformation rate $\mathbf{D}$ is defined as

$$\mathbf{D}=\frac{\partial U}{\partial\boldsymbol{\sigma}}=\dot{\bar{\varepsilon}}\,\mathbf{N},\quad \dot{\bar{\varepsilon}}=\dot{\varepsilon}_0\left(\frac{\sigma_e}{\sigma_0}\right)^n,\quad \mathbf{N}=\frac{\partial\sigma_e}{\partial\boldsymbol{\sigma}}=\frac{3}{2\,\sigma_e}\mathbf{s},\tag{4}$$

where $\mathbf{N}$ is a second order tensor of constant magnitude ($\mathbf{N}:\mathbf{N}=\frac{3}{2}$) that defines the direction of $\mathbf{D}$ and $\dot{\bar{\varepsilon}}=\sqrt{\frac{2}{3}\,\mathbf{D}:\mathbf{D}}$ is the equivalent plastic strain rate that defines the magnitude of $\mathbf{D}$. Note that $D_{kk}=0$.

The special case in which the exponent takes the value of unity $(n=1)$ corresponds to a linearly viscous solid:

$$U_L(\sigma_e)=\frac{\sigma_e^2}{6\,\mu},\quad \mathbf{D}=\frac{\partial U_L}{\partial\boldsymbol{\sigma}}=\frac{\mathbf{s}}{2\,\mu},\tag{5}$$

where $\mu=\sigma_0/(3\,\dot{\varepsilon}_0)$ is the viscosity.

The other limiting case $n\to\infty$ corresponds to a perfectly plastic solid that obeys the von Mises yield condition with flow stress