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On a tensor cross product based formulation of large strain solid mechanics

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ABSTRACT

This paper describes in detail the formulation of large strain solid mechanics based on the tensor cross product, originally presented by R. de Boer (1982) and recently re-introduced by Bonet et al. (2015a) and Bonet et al. (2015b). The paper shows how the tensor cross product facilitates the algebra associated with the area and volume maps between reference and final configurations. These maps, together with the fibre map, make up the fundamental kinematic variables in polyconvex elasticity. The algebra proposed leads to novel expressions for the tangent elastic operator which neatly separates material from geometrical dependencies. The paper derives new formulas for the spatial and material stress and their corresponding elasticity tensors. These are applied to the simple case of a Mooney–Rivlin material model. The extension to transversely isotropic material models is also considered.

1. Introduction

Large strain elastic and inelastic analysis by finite elements or other computational techniques is now well-established for many engineering applications (Belytschko et al., 2000; Bonet, 2001; Bonet et al., 2006; Bonet and Wood, 2008; Gee et al., 2009; Gil, 2006; Gil and Bonet, 2006; 2007; Gil et al., 2010; Hughes, 2000; Ortigosa et al., 2015; de Souza Neto et al., 2008; Zienkiewicz et al., 1998). Often elasticity is described by means of a hyperelastic model defined in terms of a stored energy functional which depends on the deformation gradient of the mapping between initial and final configurations (Ball, 1977; 1983; 2002; Ball and Murat, 1984; Bonet and Wood, 2008; Ciarlet, 1988; Coleman and Noll, 1959; Gonzalez and Stuart, 2008; Hill, 1957; Marsden and Hughes, 1994). It has also been shown that for the model to be well defined in a mathematical sense, this dependency with respect to the deformation gradient has to satisfy certain convexity criteria (Bonet and Wood, 2008; Gonzalez and Stuart, 2008; Marsden and Hughes, 1994). The most well-established of these criteria is the concept of polyconvexity (Ball, 1977; 1983; 2002; Ball and Murat, 1984; Ciarlet, 2010; Dacorogna, 2008; Zhang, 1992) whereby the strain energy function must be expressed as a convex function of the components of the deformation gradient, its determinant and the components of its adjoint or co-factor. Numerous authors have previously incorporated this concept into computational models for both isotropic and non-isotropic materials for a variety of applications (Kambouchev et al., 2006; Schröder, 2010; Schröder and Neff, 2003; Schröder et al., 2008; 2010; 2011).

The classical approach consists of ensuring that the stored energy function satisfies the polyconvexity condition first but then proceed towards an evaluation of stresses and elasticity tensors by re-expressing the energy function in terms of the deformation gradient alone. This inevitably leads to the differentiation of inverse functions of the deformation gradient, its transpose or the inverse of the right Cauchy–Green tensor. These derivatives are readily obtained using standard algebra but can lead to lengthy expressions. An alternative approach has recently been proposed by Bonet et al. (2015b) and Bonet et al. (2015a) by recovering the concept of the tensor cross-product originally introduced by de Boer (1982) but not previously used in continuum mechanics. This tensor cross product allows for simpler expressions to be obtained for the area and volume maps and their derivatives. The resulting formulas for the elasticity tensors provide useful physical insights by separating positive definite material components from geometrical components.

The paper explores the proposed formulation both in the reference setting, using Piola–Kirchhoff stress tensors and in the spatial setting using Kirchhoff and Cauchy stress tensors. Some formulas derived with the tensor cross product formulation are compared against their classical equivalent versions in order to demonstrate the advantages of the proposed methodology. Both

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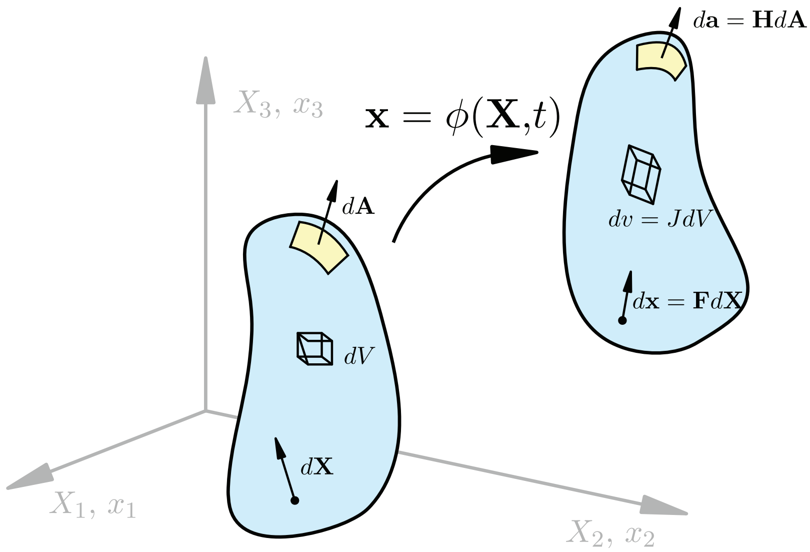

Fig. 1. Deformation mapping of a continuum and associated kinematics magnitudes: $\boldsymbol{F}$, $\boldsymbol{H}$, J.

isotropic and anisotropic cases are considered, in the latter case anisotropy is restricted to the simple transversely isotropic case. The paper illustrates the proposed concepts using the well-established model of a Mooney–Rivlin material.

The paper is organised as follows. Section 2 introduces the novel tensor cross product notation in the context of large strain deformation. Whilst this product had already been proposed by de Boer (1982) (in German), it has not previously been described in the English literature or used in the context of solid mechanics, so most readers will be unfamiliar with it. This product is used to re-express the adjoint of the deformation gradient and its directional derivatives in a novel, simple and convenient manner. Section 3 reviews the definition of polyconvex elastic strain energy functions and defines a new set of stresses conjugate to the main kinematic variables. The relationships between these stresses and the standard first Piola–Kirchhoff stresses are provided. The section also derives complementary strain energy functions in terms of the new conjugate stresses. The algebra is greatly simplified via the tensor cross product. The fourth order elasticity tensors are derived in this section taking advantage of the tensor cross product operation leading to interesting insights into the consequences of convexity. Both compressible and nearly incompressible cases are discussed in the context of Mooney–Rivlin models, although the extension to more general strain energy functions is straight forward. Section 4 derives similar equations using entirely material tensors such as the right Cauchy–Green tensor and the second Piola–Kirchhoff tensor or spatial tensors such as the Kirchhoff or Cauchy stresses. Expressions for both material and spatial elasticity tensor are given in the context of the new proposed notation. Section 5 particularises the above expressions for the case of isotropic and transversely isotropic materials. A number of mixed and complementary energy variational principles are presented in Section 6. Several of these have been used in Bonet et al. (2015b) for the purpose of constructing novel finite element approximations. Finally, Section 7 provides some concluding remarks and a summary of the key contributions of this paper.

2. Definitions and notation

2.1. Motion and deformation

Consider the three dimensional deformation of an elastic body from its initial configuration occupying a volume V, of boundary ∂V, into a final configuration at volume v, of boundary ∂v (see Fig. 1). The standard nomenclature for the deformation gradient tensor $\boldsymbol{F}$ and the Jacobian J of the deformation are used

$$d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}; \quad \boldsymbol{F} = \nabla_0\boldsymbol{x}; \tag{1a}$$

$$dv = JdV; \quad J = \det\left(\nabla_0\boldsymbol{x}\right), \tag{1b}$$

where $\boldsymbol{x}$ represents the current position of a particle originally at $\boldsymbol{X}$ and $\nabla_0 := \frac{\partial}{\partial \boldsymbol{X}}$ denotes the gradient with respect to material coordinates. Virtual or linear incremental variations of $\boldsymbol{x}$ will be denoted $\delta\boldsymbol{v}$ and $\boldsymbol{u}$, respectively. It will be assumed that $\boldsymbol{x}$ satisfy appropriate prescribed displacement based boundary conditions in $\partial_u V$, and that $\delta\boldsymbol{v}$ and $\boldsymbol{u}$ will satisfy the equivalent homogeneous conditions in this section of the boundary. Additionally, the body is under the action of certain body forces per unit undeformed volume $\boldsymbol{f}_0$ and traction per unit undeformed area $\boldsymbol{t}_0$ in $\partial_t V$, where $\partial_t V \cup \partial_u V = \partial V$ and $\partial_t V \cap \partial_u V = \emptyset$.

The element area vector is mapped from initial $d\boldsymbol{A}$ to final $d\boldsymbol{a}$ configuration by means of the two-point tensor $\boldsymbol{H}$, which is related to the deformation gradient via Nanson's rule (Bonet and Wood, 2008):

$$d\boldsymbol{a} = \boldsymbol{H}d\boldsymbol{A}; \quad \boldsymbol{H} = \det\left(\nabla_0\boldsymbol{x}\right)\left(\nabla_0\boldsymbol{x}\right)^{-T}. \tag{2}$$

Clearly, the components of this tensor are the order 2 minors of the deformation gradient and it is often referred to as the cofactor or adjoint tensor, that is $\boldsymbol{H} = \mathrm{Cof}\left(\nabla_0\boldsymbol{x}\right)$. This tensor and its derivatives will feature heavily in the formulation that follows as it is a key variable for polyconvex elastic models. Its evaluation and, more importantly, the evaluation of its derivatives using Eq. (2) is not ideal, and a more convenient formula can be derived for three dimensional applications. This relies on the use of a tensor cross product operation, presented from the first time in Ref. (de Boer, 1982), page 76, but included in 2.2 for completeness.

The relationships between $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ and the geometry $\boldsymbol{x}$ via Eqs. (1) and (2) represent three geometric compatibility conditions, which can be re-expressed in a more helpful manner via the tensor cross product defined below.

2.2. Tensor cross product

The key elements of the framework proposed is the extension of the standard vector cross product to define the cross product between second order tensors and between tensors and vectors. This rediscovers the work of de Boer (1982) which, to the best knowledge of the authors, does not appear in any English language publication. The original nomenclature in de Boer (1982) is "Das äuβere