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A generalized theory of elastodynamic homogenization for periodic media

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ABSTRACT

For periodically inhomogeneous media, a generalized theory of elastodynamic homogenization is proposed so that even the long-wavelength and low-frequency asymptotic expansions of the resulting effective (or macroscopic) motion equation can, approximately but simultaneously, capture all the acoustic and some of the optical branches of the microscopic dispersion curve. The key to constructing the generalized theory resides in incorporating new kinematical degrees of freedom in conjunction with rapidly oscillating body forces as microscopic and macroscopic loadings while satisfying an energetical consistency constraint reminiscent of Hill–Mandel lemma. By this constraint, an effective displacement field is naturally defined as the projection of a microscopic one onto the dual to the space of body forces. To illustrate these results, a two-phase string is studied in detail.

1. Introduction

The elastodynamic homogenization approaches reported up to now in the literature are observed to run into difficulties when being used to model dynamical effects over a wide frequency range.

1. The classical lowest-order Long-Wavelength (LW) Low-Frequency (LF) homogenization approaches (Bensoussan et al., 1978; Sanchez-Palencia, 1980) yield a homogeneous substitution Cauchy medium which misses all dispersive effects and all internal resonances, i.e., all optical oscillation modes.
2. The higher-order LW-LF asymptotic homogenization approaches (Andrianov et al., 2008; Boutin and Auriault, 1993) lead to effective strain-gradient media which can model well dispersive behaviors and size effects but are valid only near the acoustic branches independently of the order of the asymptotic approximations used.
3. The high-frequency asymptotic approaches (Antonakakis et al., 2014; Boutin et al., 2014; Colquitt et al., 2014; Craster et al., 2010; Daya et al., 2002; Nolde et al., 2011) are successful in capturing high-frequency optical modes but still valid only in the vicinity of some finite frequency.
4. The high-contrast asymptotic approaches (Auriault and Bonnet, 1985; Auriault and Boutin, 2012; Smyshlyaev, 2009) have a wide frequency validity domain englobing an infinite number of optical branches. However, the corresponding effective behavior is complex and nonlocal in time.
5. The non-asymptotic theory of Willis (1997, 2011) yields exactly the whole dispersion curve. Nonetheless, the described effective fields are only relevant for low frequencies (Nassar et al., 2015b; Srivastava and Nemat-Nasser, 2014).

The main purpose of the present paper is to construct a generalized theory of elastodynamic homogenization for periodic media which improves the quality of the Willis effective behavior as an approximation to the microscopic behavior in a way that LW-LF asymptotic expansions become able to capture, approximately but simultaneously, all the acoustic and some of the optical branches of the microscopic dispersion curve. To achieve this purpose, new kinematical Degrees Of Freedom (DOFs) are taken into account so as to describe some short-wavelength components of the microscopic displacement field which become dominant at high frequencies. The new DOFs are excited by incorporating various rapidly oscillating body forces on the microscale and on the macroscale under an energetical consistency constraint hereafter called Energy Equivalency Principle (EEP). The EEP is a balance between the microscopic and macroscopic virtual works and is later proven to yield a generalized version of the well-known Hill–Mandel lemma. With respect to Willis theory, we underline two major differences. First, the incorporated loadings are much richer than those employed by Willis (1997, 2011). This has the consequence of reducing the error committed during the upscaling process and providing an extended frequency validity domain. Second, the EEP concerns virtual works and not their expectancies. From the physical

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standpoint, this leads to a clear distinction between the macroscale and the microscale in terms of wavelengths. Nevertheless, it should be pointed out that the generalized theory presented here is by construction limited to periodically inhomogeneous media while Willis theory is formally valid both for periodically and randomly inhomogeneous media.

The paper is organized as follows. In Section 2, we recall some geometrical elements useful for describing periodic media, summarize the equations governing the kinematics and dynamics of them, and simplify these equations by using Bloch-wave expansions. The main body of the generalized theory is presented in Section 3. The EEP is first postulated; the space of admissible body forces is then defined as the set of macroscopically applied loadings; the effective displacement field associated to a microscopic displacement is obtained by the EEP and proven to be an improvement over the one defined by Willis; the effective motion equation is finally derived in a formal way and a Hil–Mandel relation is demonstrated. In Section 4, an analytical LW-LF asymptotic approximation to the effective motion equation is given for a particular 1D two-phase string. Exact and approximate dispersion curves are plotted and compared. It appears then how the resulting asymptotic model, though based on LF expansions, can simultaneously capture acoustic and optical branches while conserving a low-order local motion equation.

2. Preliminaries

In this section, some geometrical elements useful for the study of periodic media are recalled. The governing equations of linear elasticity are recapitulated. Bloch-wave expansions of fields and work are also introduced.

2.1. Geometry and periodicity

Let Ω be a d-dimensional infinite body. Define $\mathscr{E}$ as the vector space of translations acting on the points of Ω. Given d independent translations $(\boldsymbol{b}_j)_{j=1\dots d}$, denote by $\mathscr{R}$ the subset of $\mathscr{E}$ obtained by integer combinations of these vectors. The subset $\mathscr{R}$ is called a lattice. Then, a scalar, vector or tensor field h defined over Ω is said to be $\mathscr{R}$-periodic if and only if it satisfies $h(\boldsymbol{x}+\boldsymbol{r}) = h(\boldsymbol{x})$ for all points $\boldsymbol{x} \in \Omega$ and all translations $\boldsymbol{r} \in \mathscr{R}$. Accordingly, h needs being defined only over a unit cell

$$T = \left\{ \boldsymbol{x}_o + \boldsymbol{r} \middle| \boldsymbol{r} = \sum_{j=1}^{d} r_j \boldsymbol{b}_j, -1/2 \le r_j < 1/2 \right\} \subset \Omega,$$

where $\boldsymbol{x}_o$, its center, is an arbitrary point of Ω. Note that while $\mathscr{R}$-periodicity is well defined, the choice of $\boldsymbol{b}_j$ and T is not unique.

Symbolize by $\mathscr{E}^*$ the dual space of $\mathscr{E}$. A wavenumber $\boldsymbol{k} \in \mathscr{E}^*$ acting on a translation $\boldsymbol{r} \in \mathscr{E}$ produces a phase shift $\boldsymbol{k} \cdot \boldsymbol{r}$ where $(\cdot)$ is the usual dot product. Now, points of Ω and vectors of $\mathscr{E}$ can be identified after choosing some origin $\boldsymbol{x}_o$. In what follows, we drop $\boldsymbol{x}_o$ so as to write $\boldsymbol{k} \cdot \boldsymbol{x}$ instead of $\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}_o)$ for simplicity. The reciprocal lattice $\mathscr{R}^*$ of the direct lattice $\mathscr{R}$ is defined as the subset of $\mathscr{E}^*$ consisting of wavenumbers $\boldsymbol{\xi}$ such that $e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}$ is $\mathscr{R}$-periodic, with $i^2 = -1$. Also of interest is the first Brillouin zone T^* defined as the set of wavenumbers closer to the null wavenumber than to any other wavenumber of $\mathscr{R}^*$, i.e.,

$$T^* = \left\{ \boldsymbol{k} \in \mathscr{E}^* | \|\boldsymbol{k}\| < \|\boldsymbol{k} - \boldsymbol{\xi}\|, \forall \boldsymbol{\xi} \in \mathscr{R}^* - \{\boldsymbol{0}\} \right\}.$$

This zone is uniquely defined and independent of T.

A function h defined over Ω can be expanded into plane waves over $\mathscr{E}^*$ such that

$$h(\boldsymbol{x}) = \int_{\mathscr{E}^*} \tilde{h}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}^d \boldsymbol{k}.$$

In particular, when h is $\mathscr{R}$-periodic, it can be written as the Fourier series

$$h(\boldsymbol{x}) = \sum_{\boldsymbol{\xi} \in \mathscr{R}^*} \tilde{h}_{\boldsymbol{\xi}} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}.$$

Having this in mind, with respect to $\mathscr{R}$, T^* can be seen as the support of slowly varying fields. In particular, among $\mathscr{R}$-periodic functions, only constants have their wavenumber contained in T^*, i.e., $T^* \cap \mathscr{R}^* = \{\boldsymbol{0}\}$.

Finally, call a Bloch wave, of wavenumber $\boldsymbol{k}$ and amplitude $\tilde{h}_{\boldsymbol{k}}(\boldsymbol{x})$, a function $h_{\boldsymbol{k}}(\boldsymbol{x})$ of the form

$$h_{\boldsymbol{k}}(\boldsymbol{x}) = \tilde{h}_{\boldsymbol{k}}(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $\tilde{h}_{\boldsymbol{k}}(\boldsymbol{x})$ is $\mathscr{R}$-periodic.

2.2. Constitutive and motion equations

Letting $\boldsymbol{u}(\boldsymbol{x}, t)$ be the displacement vector for a point $\boldsymbol{x} \in \Omega$ at instant t, the strain field $\boldsymbol{\varepsilon}$ and velocity field $\boldsymbol{v}$ are derived according to

$$\boldsymbol{\varepsilon} = \nabla \otimes^s \boldsymbol{u}, \quad \boldsymbol{v} = \dot{\boldsymbol{u}},$$

where ∇ is the space gradient operator, $\otimes$ denotes the tensor product, the superscripted "s" indicates symmetrization and a superscripted dot symbolizes differentiation with respect to time. The stress tensor $\boldsymbol{\sigma}$ and momentum density $\boldsymbol{p}$ are then given by the local constitutive equations of Ω:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}, \quad \boldsymbol{p} = \rho \boldsymbol{v},$$

with $\boldsymbol{C}$ and ρ being the elastic stiffness tensor and the scalar mass density, respectively, and the colon (:) standing for double contraction.

The motion equation of Ω reads

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \dot{\boldsymbol{p}}$$

where $(\nabla \cdot)$ is the divergence operator and $\boldsymbol{f}$ is a field of externally applied body forces. We shall mostly work with harmonic fields of frequency ω. Therefore, all time derivatives can be substituted by $i\omega$-multiplications and time dependency can be dropped henceforth. The motion equation of Ω becomes the Helmholtz equation

$$\nabla \cdot \left[\boldsymbol{C}(\boldsymbol{x}) : \left(\nabla \otimes^s \boldsymbol{u}(\boldsymbol{x}) \right) \right] + \boldsymbol{f}(\boldsymbol{x}) = -\omega^2 \rho(\boldsymbol{x}) \boldsymbol{u}(\boldsymbol{x}) \tag{2.1}$$

where we have displayed $\boldsymbol{x}$-dependencies and omitted ω-dependencies.

In this work, the homogenization of Ω amounts to finding the motion equation, hereafter called "effective motion equation", of a homogeneous medium substituting the initial inhomogeneous one, under an energy equivalency constraint to be specified.

2.3. Bloch-wave expansions

The superposition principle makes it possible to work with elementary, such as plane-wave, body forces instead of arbitrary ones $\boldsymbol{f}(\boldsymbol{x})$. It is however more convenient, for reasons that will become clear, to work with Bloch-wave body forces. Then, let $\boldsymbol{f}_{\boldsymbol{k}}(\boldsymbol{x})$ be an element of the Bloch-wave expansion of $\boldsymbol{f}(\boldsymbol{x})$ such that

$$\boldsymbol{f}(\boldsymbol{x}) = \int_{T^*} \boldsymbol{f}_{\boldsymbol{k}}(\boldsymbol{x}) \, \mathrm{d}^d \boldsymbol{k} \equiv \int_{T^*} \tilde{\boldsymbol{f}}_{\boldsymbol{k}}(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}^d \boldsymbol{k}, \tag{2.2}$$

where $\tilde{\boldsymbol{f}}_{\boldsymbol{k}}(\boldsymbol{x})$ is $\mathscr{R}$-periodic and the symbol $\equiv$ stands for equality by definition.

For a given $\boldsymbol{k} \in T^*$, the motion equation for a Bloch-wave body force takes the form

$$\nabla \cdot \left[\boldsymbol{C}(\boldsymbol{x}) : \left(\nabla \otimes^s \boldsymbol{u}_{\boldsymbol{k}}(\boldsymbol{x}) \right) \right] + \tilde{\boldsymbol{f}}_{\boldsymbol{k}}(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = -\omega^2 \rho(\boldsymbol{x}) \boldsymbol{u}_{\boldsymbol{k}}(\boldsymbol{x}).$$