International Journal of Solids and Structures 81 (2016) 294–310

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

A high order model for piezoelectric rods: An asymptotic approach

J.M. Viaño[a], C. Ribeiro[b], J. Figueiredo[b], Á. Rodríguez-Arós[c,*]

[a] Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Spain
[b] Departamento de Matemática e Aplicações and Centro de Matemática, Universidade do Minho, Portugal
[c] Departamento de Métodos Matemáticos e Representación, Universidade da Coruña, Spain

ARTICLE INFO

Article history:
Received 21 March 2014
Revised 30 October 2015
Available online 11 December 2015

Keywords:
Asymptotic analysis
Elastic rod
Beams
Piezoelectric material
Anisotropy

ABSTRACT

This paper is devoted to the study of an electromechanical model for a linear transversely inhomogeneous anisotropic rod made of piezoelectric symmetry class 2 materials. The materials in this class include, among many others, Barium Titanate (BT), Lead Zirconium Titanate (PZT) and Polyvinylidene Fluoride (PVDF).

For that purpose, we use asymptotic analysis in order to derive from the 3D piezoelectricity problem a reduced model, without making any *a priori* assumptions of geometrical or mechanical nature. This process involves considering the diameter of the cross section h as *small parameter*, the assumption of existence of asymptotic expansions of the unknowns (displacements field and electric potential) and the characterization of their respective leading terms as the unique solutions of limit reduced coupled equations. Finally, the validity of the limit reduced model is supported by providing rigorous convergence results.

1. Introduction

The so-called "intelligent structures", i.e. those that are able to adapt in an automatic way when submitted to external actions in order to provide an optimal behavior, play an increasing role in the framework of nowadays security and wellbeing tools, and their importance will certainly increase in the near future. The use of piezoelectric materials is a strategy that has been used with considerable success in the development of new materials and technologies for the production of these structures.

The piezoelectric effect is an electromechanical property that is exhibited by many materials: when the material is deformed by the action of external forces or pressure, an electric field appears – direct piezoelectric effect – and conversely, when subject to an electric potential, the material suffers a deformation – inverse piezoelectric effect. This reciprocity provides both an efficient sensor capacity and the possibility of acting over large portions of structures without a considerable weight increase. For more complete information on the properties and use of piezoelectric materials see, for example, Banks et al. (1996); Ikeda (1990); Nye (1985); Royer and Dieulesaint (2000); Taylor et al. (1985). More specifically, information about traditional piezoelectric beam theories can be consulted in Bellis and Imperiale (2014); Bisegna (1998); Rovenski and Abramovich (2009).

From the above considerations one easily concludes on the relevance of designing new multifunctional materials (e.g. piezoelectric materials) and, consequently, on the importance of accurately knowing the electromechanical behavior of structures in which these materials are applied. Some important results in this field were successfully obtained by applying two different modeling techniques: homogenization and asymptotic analysis (see Cagnol et al., 2008; Miara et al., 2007 for a review of the state of the art). See in particular, Castillero et al. (1998); Ghergu et al. (2007); Mechkour (2004); Miara et al. (2005) for homogenization of piezoelectric plates and shells and Collard and Miara (2002); Figueiredo and Leal (2006); Rahmoune et al. (1998); Sabu (2002); Sène (2001); Weller and Licht (2002); Weller and Licht (2004) for formal asymptotic analysis to justify lower dimensional constitutive laws for piezoelectric plates, shells and rods. The above cited works were based on relevant and pioneer work in applying homogenization and asymptotic analysis techniques in elasticity problems (linear and nonlinear). See in particular Ciarlet (1980); Collard and Miara (1997); Le Dret and Raoult (1995); Lewiński and Telega (2000); Miara (1994); Sanchez-Hubert and Sanchez-Palencia (1997) for plates, Bermúdez and Viaño (1984a); Cimetière et al. (1988); Mascarenhas and Trabucho (1990); Trabucho and Viaño (1996); Tutek (1987); Tutek and Aganović (1986) for rods, and Caillerie and Sanchez-Palencia (1995); Ciarlet (2000); Ciarlet et al. (1996); Pitkaranta and Sanchez-Palencia (1997) for shells.

As it is well known, the classical theories of thin structures obtained from classical formulations are intuitive for understanding, but the *a priori* assumptions are often mechanically unjustified and can reveal inadequate for the analysis of real structures if those

* Corresponding author. Tel.: +34 981167000.
E-mail addresses: juan.viano@usc.es (J.M. Viaño), cribeiro@math.uminho.pt (C. Ribeiro), jmfiguei@math.uminho.pt (J. Figueiredo), angel.aros@udc.es (Á. Rodríguez-Arós).

http://dx.doi.org/10.1016/j.ijsolstr.2015.12.005
0020-7683/

a priori assumptions are not met. Although the amount of research in the asymptotic analysis field is quite large, only a small number of works exist (as far as we know) devoted to the derivation of models for piezoelectric rods (see Figueiredo and Leal, 2006; Weller and Licht, 2004). Despite the progress made in those papers in order to derive reduced models for piezoelectric beams, the subject is still open to further research.

In this framework, our contribution consists in the obtention of a reduced model of a linear transversely inhomogeneous anisotropic rod made of piezoelectric symmetry class 2 materials, by using the asymptotic method introduced by Lions (see Lions, 1973), taking the diameter of the rod cross section h as small parameter, without any *a priori* hypothesis of geometrical or mechanical nature. The model obtained in this way allows to, simultaneously, justify and/or complement the traditional piezoelectric beam theories (such as those of Euler–Bernoulli, Timoshenko, and Vlasov). Additionally, in the reduced model, to our knowledge for the first time, the electric potential is coupled with first order terms of the asymptotic expansion of the displacements.

The present work is organized in the following manner. In Section 2 we show the 3D piezoelectric problem posed in its domain of reference Ω^h (the details on the notation used will be specified in the corresponding sections and can be found in tables in the appendix as well) and formulate it in the form of the virtual work principle (see (2.13)):

$$\text{Find } (\boldsymbol{u}^h,\varphi^h)\in \boldsymbol{V}^h(\Omega^h)\times H^1(\Omega^h),\quad \text{such that } \varphi^h=\hat{\varphi}^h \text{ on } \Gamma^h_{eD},$$
$$\int_{\Omega^h}\sigma^h_{ij}(\boldsymbol{u}^h,\varphi^h)e^h_{ij}(\boldsymbol{v}^h)d\boldsymbol{x}^h+\int_{\Omega^h}D^h_k(\boldsymbol{u}^h,\varphi^h)E^h_k(\psi^h)d\boldsymbol{x}^h$$
$$=\int_{\Omega^h}f^h_iv^h_id\boldsymbol{x}^h+\int_{\Gamma^h_N}g^h_iv^h_id\Gamma^h,$$
$$\text{for all } (\boldsymbol{v}^h,\psi^h)\in \boldsymbol{V}^h(\Omega^h)\times\Psi^h(\Omega^h),$$

where $\boldsymbol{u}^h=(u^h_i)$ and φ^h are the displacements field and the electric potential, respectively, to be found in the corresponding admissible spaces $\boldsymbol{V}^h(\Omega^h)$ and $H^1(\Omega^h)$. Also, $\boldsymbol{\sigma}^h=(\sigma^h_{ij})$ and $\boldsymbol{D}^h=(D^h_i)$ represent the stress tensor field and the electric displacements field, respectively, while $\boldsymbol{e}^h(\boldsymbol{v}^h)=(e^h_{ij}(\boldsymbol{v}^h))=(\frac{1}{2}(\frac{\partial v^h_i}{\partial x^h_j}+\frac{\partial v^h_j}{\partial x^h_i}))$ and $\boldsymbol{E}^h=(E^h_i(\psi^h))=(-\frac{\partial\psi^h}{\partial x^h_i})$ are the deformation operator and minus the gradient operator. The constitutive equations are (see (2.3)):

$$\begin{cases}\sigma^h_{ij}(\boldsymbol{u}^h,\varphi^h)=C^h_{ijkl}e^h_{kl}(\boldsymbol{u}^h)-P^h_{kij}E^h_k(\varphi^h), & \text{in } \Omega^h,\\ D^h_i(\boldsymbol{u}^h,\varphi^h)=P^h_{ikl}e^h_{kl}(\boldsymbol{u}^h)+\varepsilon^h_{ij}E^h_j(\varphi^h), & \text{in } \Omega^h,\end{cases}$$

where $\boldsymbol{C}^h=(C^h_{ijkl})$, $\boldsymbol{P}^h=(P^h_{kij})$ and $\boldsymbol{\varepsilon}^h=(\varepsilon^h_{ij})$ are the stiffness, piezoelectric and dielectric tensors, respectively.

In Section 3, we perform a change of variable and a scaling of the unknowns (to obtain the displacements field $\boldsymbol{u}(h)$ and the electric potential $\varphi(h)$) and data, so that the virtual work formulation is now posed in a domain independent on h, denoted by Ω.

In Section 4, we assume that the scaled unknowns can be written in the form of asymptotic developments, allowing to identify and characterize the corresponding leading terms; this procedure yields the new reduced model, which consists on a set of 1D variational problems for the displacements field and a variational problem for the electric potential and the first order term of the displacement, where some terms are not local.

The strong convergence of the unknowns as the small parameter tends to zero is proved in Section 5. This is a key step in order to give a rigorous mathematical justification of the reduced models of Section 4.

In Section 6 we undo the change of variable and de-scale the quantities appearing in the reduced model and constitutive equations derived in Section 4 in order to express both in the original reference domain Ω^h and thus have a true physical meaning. We find that the asymptotic expansion of the true displacements field $\boldsymbol{u}^h=(u^h_i)$ is of the form:

$$u^h_\alpha=\xi^h_\alpha+u^{1h}_\alpha+O(h),\quad u^h_3(\boldsymbol{x}^h)=\xi^h_3-x^h_\beta\partial^h_3\xi^h_\beta+u^{1h}_3+O(h^2),$$

where the leading term $\boldsymbol{u}^{0h}=(u^{0h}_i)$ is characterized by functions $\xi^h_\alpha\in H^2_0(0,L)$, $\xi^h_3\in H^1_0(0,L)$, satisfying the purely mechanical stretching and beam equations (see (6.5) and (6.6)):

$$\xi^h_\alpha\in H^2_0(0,L),\quad \int_0^L YI^h_\alpha\partial^h_{33}\xi^h_\alpha\partial^h_{33}\chi^h_\alpha dx^h_3$$
$$=\int_0^L F^h_\alpha\chi^h_\alpha dx^h_3-\int_0^L M^h_\alpha\partial^h_3\chi^h_\alpha dx^h_3,\quad (\text{no sum on }\alpha),$$
$$\text{for all}\quad \chi^h_\alpha\in H^2_0(0,L),$$
$$\xi^h_3\in H^1_0(0,L),\quad \int_0^L YA(\omega^h)\partial^h_3\xi^h_3\partial^h_3\chi^h_3dx^h_3$$
$$=\int_0^L F^h_3\chi^h_3dx^h_3,\ \text{for all } \chi^h_3\in H^1_0(0,L).$$

On the other hand, $\boldsymbol{u}^{1h}=(u^{1h}_i)$, the following order term of $\boldsymbol{u}^h$, is of the following form:

$$u^{1h}_\alpha=z^h_\alpha+\delta^h_\alpha z^h,\quad u^{1h}_3=-r^h-z^h_3-x^h_\alpha(z^h_\alpha)'-w^h(z^h)',$$

where $w^h\in R(\Omega)$ is the warping function, depending only on the geometry and stiffness tensor, while $z^h_\alpha, z^h\in H^1_0(0,L)$, $z^h_3\in L^2(0,L)$, and $r^h\in R(\Omega)$ is a function depending on the electric potential, thus giving the coupling between the mechanical and electrical quantities. More specifically, we find that the asymptotic expansion of the true electric potential φ^h is

$$\varphi^h=\hat{\varphi}^{0h}+\bar{\varphi}^{0h}+O(h^2),$$

where $\bar{\varphi}^{0h}\in\Psi_{l0}(\Omega)$ and the coupled equations for z^h, r^h and $\bar{\varphi}^{0h}$ are given by (see (6.8)–(6.10)):

$$\int_0^L J^h\partial^h_3z^h\partial^h_3\zeta\, dx_3-\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\left(\partial^h_\alpha w^h-\delta^h_\alpha\right)\partial^h_\beta\bar{\varphi}^{0h}\,d\hat{\boldsymbol{x}}^h\right)\partial^h_3\zeta^h\, dx_3$$
$$=\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial^h_\beta\hat{\varphi}^h\left(\partial^h_\alpha w^h-\delta^h_\alpha\right)d\hat{\boldsymbol{x}}^h\right)\partial^h_3\zeta^h\, dx_3\quad\forall\zeta^h\in H^1_0(0,L),$$
$$\int_0^L\left(\int_{\omega^h}C^h_{3\alpha3\beta}\partial^h_\beta r^h\partial^h_\alpha\rho^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\rho^h_3\, dx_3$$
$$-\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial_\beta\bar{\varphi}^{0h}\partial_\alpha\rho^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\rho^h_3\, dx_3$$
$$=\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial_\beta\hat{\varphi}^h\partial_\alpha\rho^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\rho^h_3\, dx_3\ \forall\rho^h_\sharp\in Q^h(\omega^h),$$
$$\times\rho^h_3\in L^2(0,L),$$
$$\int_0^L\left(\int_{\omega^h}\varepsilon^h_{\alpha\beta}\partial^h_\beta\bar{\varphi}^{0h}\partial^h_\alpha\psi^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\psi^h_3\, dx_3$$
$$+\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial^h_\alpha r^h\partial^h_\beta\psi^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\psi^h_3\, dx_3$$
$$+\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\left(\partial^h_\alpha w^h-\delta^h_\alpha\right)\partial^h_\beta\psi^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\partial^h_3z^h\psi^h_3\, dx_3$$
$$=-\int_0^L\left(\int_{\omega^h}\varepsilon^h_{\alpha\beta}\partial^h_\beta\hat{\varphi}^h\partial^h_\alpha\psi^h_\sharp\, d\hat{\boldsymbol{x}}^h\right)\psi^h_3\, dx_3\ \forall\psi^h_\sharp\in S(\omega^h),$$
$$\times\psi^h_3\in L^2(0,L).$$

Notice that even though we know the existence of z^h_α and z^h_3, these functions are not characterized at this stage. Nevertheless, they are not needed in order to obtain φ^{0h}, which depends only on $\boldsymbol{u}^{1h}$ through r^h.

We then briefly comment on some particular cases, specifically when the cross-section is square, which leads to some practical simplifications, and we finish by giving some conclusions.