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Estimating the non-homogeneous elastic modulus distribution from surface deformations

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ABSTRACT

We present a novel approach to solve the inverse problem in finite elasticity for the non-homogeneous shear modulus distribution solely from known surface deformation fields. The inverse problem is posed as a constrained optimization problem under regularization and solved utilizing the adjoint equations. Hypothetical "measured" surface displacement fields are created, by inducing indentations on the exterior of the specimen. These surface displacement fields are used to test the inverse strategy on a problem domain consisting of a stiff circular inclusion in a softer homogeneous background. We observe that the shear modulus reconstruction as well as the shape of the circular inclusion improves with an increasing number of surface displacement fields. Furthermore, with increasing noise level in the surface displacement field, the contrast of the reconstructions decreases.

1. Introduction

Inverse problems in elasticity aim to solve for the non-homogeneous material properties, requiring the knowledge of displacement fields and boundary conditions, e.g. Neumann and Dirichlet boundary conditions. Material properties, such as the Young's modulus, shear modulus, or Poisson's ratio can be determined for a linear elastic material non-destructively. Other constitutive models, such as viscoelastic, poro-elastic, and hyperelastic models, will possess additional material parameters and can potentially be determined from dynamic displacement and large displacement data (Goenezen, 2011; Goenezen et al., 2011a, 2011b; Ipek-Ugay et al., 2015; Tzschätzsch et al., 2015). Knowing the material parameter distribution in a solid could potentially have important applications in a broad range of disciplines. In human or animal tissue mechanics the change in material properties could be correlated to distinct tissue types, thus could help to classify tissues non-invasively and to monitor their change periodically in the living organism. Quantifying the material properties of diseased tissues may provide clues on the type of the disease (Samani and Plewes, 2007a, 2007b; O'Hagan and Samani, 2009; Goenezen, 2011; Richards and Doyley, 2011; Goenezen et al., 2012). This can be done as the tissue composition changes with disease development, which is manifested in the macroscopic mechanical response. Another potential application area is in detecting structural failure by monitoring changes in their local mechanical response.

As mentioned earlier, the solution of the inverse problem requires measured displacement fields, which in general are acquired in the entire interior of the solid. This is currently done using state of the art magnetic resonance imaging (Muthupillai et al., 1995; Sack et al., 2001; Shah et al., 2004; Othman et al., 2005; Atay et al., 2008; Kwon et al., 2009; Neu et al., 2009; Othman et al., 2012; Johnson et al., 2013), ultrasound techniques (Ophir et al. 1991, Varghese and Ophir 1996, Garra et al. 1997, de Korte et al. 1998, Ophir et al. 1999, Varghese et al. 2000, Burnside et al. 2007, Patil et al. 2008), optical coherence tomography (Schmitt 1998, Peng et al. 2011, Goenezen et al. 2012, Nguyen et al. 2014), or computed tomography scans. This is feasible as these devices enable to image the interior of solids while being actively or passively deformed. Furthermore, these imaging modalities are non-invasive, thus the material properties of a non-homogeneous tissue can be determined *non-invasively* and *in vivo*. This opens up new possibilities in diagnostic imaging modalities to detect and diagnose diseased tissues such as breast cancers, liver cirrhosis, malignant melanoma, prostate cancer, etc. The inverse problem has been solved directly from the underlying partial differential equation in elasticity (Skovoroda et al., 1999; Zhu et al., 2003) and indirectly by minimizing an objective function under the constraint of the underlying partial differential equations in elasticity (Kallel and Bertrand, 1996; Doyley et al., 2000; Gokhale et al., 2008; Oberai et al., 2009; Goenezen, 2011; Goenezen et al., 2011a, 2011b, 2012; Richards and Doyley, 2011; Barbone et al., 2014).

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An alternative approach to solve the inverse problem in elasticity is by using observations from the exterior of the sample only. This procedure provides little information to infer the subsurface material property distribution of the sample. The solution to this problem has been shown with hypothetical data (Schnur and Zabaras, 1992; Liu et al., 2003; Peters et al., 2004; Olson and Throne, 2010, 2013) and experimental data (Samani et al., 2003, 2007; Samani and Plewes 2004, 2007a, 2007b; Peters et al. 2006, 2008a, 2008b, 2009; O'Hagan and Samani 2008; Lotz et al. 2010; Van Houten et al., 2011; Kaster et al. 2011; Omidi et al. 2014) to recover the target material property distribution using limited surface observations. Liu et al. (2003) developed an analytic method based on micromechanical theory while other techniques rely on finite element methods (Schnur and Zabaras, 1992; Samani et al., 2003, 2007; Peters et al., 2004, 2006, 2008a, 2008b, 2009; Samani and Plewes 2004, 2007; O'Hagan and Samani 2008; Lotz et al., 2010; Olson and Throne, 2010, 2013; Van Houten et al., 2011; Kaster et al. 2011; Omidi et al., 2014). These approaches utilized measured static surface deformations (Schnur and Zabaras, 1992; Liu et al., 2003; Samani et al., 2003, 2007; Samani and Plewes, 2004, 2007; O'Hagan and Samani, 2008; Olson and Throne 2010, 2013; Kaster et al. 2011; Omidi et al. 2014), time harmonic surface deformations using high-speed cameras (Peters et al., 2004, 2006, 2008, 2009; Lotz et al., 2010; Van Houten et al., 2011), and force information (Samani et al., 2003, 2007; Samani and Plewes, 2004, 2007; O'Hagan and Samani, 2008; Kaster et al., 2011; Omidi et al. 2014). In Samani and Plewes (2004), O'Hagan and Samani (2008), Kaster et al. (2011), and Omidi et al. (2014) larger deformations were acquired to quantify the nonlinear material response for various hyperelastic models on excised breast and brain tissues. Therein, the authors did not intend to develop a non-invasive approach, but rather have shown distinct nonlinear responses for diseased tissues in general. The approaches discussed above utilize a priori information that in general may not be available. Assumptions such as knowing the location of the object in the solid domain, its size, or shape, limit their practical application. Furthermore, the solution space is equal to the number of unknown properties of the objects and their surrounding material.

In this paper, we will solve the inverse problem in elasticity in a hypothetical study for the shear modulus distribution using only surface deformations. This methodology does not require any a priori information about the problem domain. It is based on finite element techniques, and the shear modulus distribution is represented as unknowns on the mesh nodes and interpolated with finite element shape functions. Thus, the number of unknown shear modulus values is equal to the total number of finite element nodes. We will test this method on a problem domain consisting of an inclusion embedded in a homogeneous background, and recover the shear modulus distribution using simulated surface displacement fields. Additionally, we add noise to the data to mimic measured surface deformations from recorded digital camera images.

2. Methods

2.1. Inverse problem formulation utilizing measured surface displacement

The solution of the inverse problem is highly unstable to noisy displacement data, thus is formulated as a constrained optimization problem. More precisely, the misfits between measured and computed surface displacements are minimized under the "control" of a regularization term. One "natural" way to formulate the inverse problem statement is as follows. Find the shear modulus distribution μ such that the objective function

$$F = \sum_{i=1}^{n} \int_{\Gamma_i} \left(\mathbf{u}^i - \mathbf{u}^i_{\mathrm{meas}}\right)^2 d\Gamma + \alpha \mathrm{Reg}(\mu) \tag{1.1}$$

is minimized under the constraint of the forward elasticity problem. The first term is the displacement correlation term, minimizing the square of the misfit between the computed $\mathbf{u}^i$ and measured $\mathbf{u}^i_{\mathrm{meas}}$ surface deformations on the problem boundary. The summation indicates that this formulation can accommodate multiple surface displacement fields, where n denotes the total number of observations. It is emphasized that the boundary integral Γ_i is intentionally augmented with the index i to accommodate surface displacement data on varying boundaries. This is because it may not be feasible to observe data on the same sub-boundary domain for each experiment. The second term is the so-called regularization term to penalize oscillations in the final solution of the shear modulus distribution. We will define the particular regularization type later on.

The inverse problem formulation in Eq. (1.1) is expressed analogous to Kallel and Bertrand (1996), Doyley et al. (2000), Oberai et al. (2003), Gokhale et al. (2008), Goenezen (2011), Goenezen et al. (2011a, 2011b, 2012), Hall et al. (2011), Mei and Goenezen (2015), but differs in that the predicted and measured displacements are correlated on the surface while the referenced approaches correlate the displacements in the entire interior of the problem domain. Discretizing Eq. (1.1) with finite element techniques is straightforward. This will be demonstrated for one displacement field to reduce notations, this is

$$F = \int_{\Gamma} (\Delta \mathbf{u})^2 d\Gamma + \alpha \mathrm{Reg}(\mu), \tag{1.2}$$

where $\Delta \mathbf{u} = \mathbf{u} - \mathbf{u}_{\mathrm{meas}}$. The finite element interpolation yields

$$F = \sum_{e=1}^{N} \int_{\Gamma_e} \left[\sum_{j=1}^{N_e} \Delta \mathbf{u}^e_j N^e_j(\mathbf{x}) \right]^2 d\Gamma + \alpha \mathrm{Reg}\left(\sum_{k=1}^{N_n} \mu_j N_j(\mathbf{x}) \right) \tag{1.3}$$

where N, N_e, and N_n denote the total number of finite elements on the boundary, the number of nodes on each element, and the total number of mesh nodes in the problem domain, respectively. Further, $N^e_j(\mathbf{x})$ denotes the shape function for the jth node corresponding to the eth linear triangular element. While this approach appears to be reasonable, we employed an alternative formulation to facilitate implementation. More precisely, we have used domain integrals over finite elements at the boundary, given by

$$F = \sum_{e=1}^{\bar{N}} \int_{\Omega_e} \left[\sum_{j=1}^{\bar{N}_e} \Delta \mathbf{u}^e_j N^e_j(\mathbf{x}) \right]^2 d\Omega + \alpha \mathrm{Reg}\left(\sum_{k=1}^{N_n} \mu_j N_j(\mathbf{x}) \right) \tag{1.4}$$

where $\bar{N}$ denotes the total number of domain elements at the boundary and $\bar{N}_e$ denotes the number of element nodes on the boundary of the corresponding elements. It is noted that only displacement information on the boundaries is assumed to be known, despite the integration over element domains. This more "unnatural" approach has been performed to use the current framework of the existing inverse solver written for minimizing the misfit in displacements over the volume integral. In the following we will analyze the implications of using Eq. (1.3) versus Eq. (1.4) on a uniform mesh given in Fig. 1. The coordinate s spans along one boundary edge of a problem domain in two dimensional space and we discard the other boundary edges to simplify the analysis (see bold elements in Fig. 1). The width of the elements along the t coordinate is denoted by a, and the height along the s coordinate of the elements is denoted by b. Evaluating Eq. (1.4) for the boundary