International Journal of Solids and Structures 83 (2016) 126–141

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

# Simplified modelling of chiral lattice materials with local resonators

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ARTICLE INFO

*Article history:*
Received 6 August 2015
Revised 30 December 2015
Available online 19 January 2016

*Keywords:*
Chiral lattice
Metamaterials
Local resonator
Band gaps
Floquet–Bloch spectrum
Dispersive waves

ABSTRACT

A simplified model of periodic chiral beam-lattices containing local resonators has been formulated to obtain a better understanding of the influence of the chirality and of the dynamic characteristics of the local resonators on the acoustic behaviour. The beam-lattice models are made up of a periodic array of rigid heavy rings, each one connected to the others through elastic slender massless ligaments and containing an internal resonator made of a rigid disk in a soft elastic annulus. The band structure and the occurrence of low frequency band-gaps are analysed through a discrete Lagrangian model. For both the hexa- and the tetrachiral lattice, two acoustic modes and four optical modes are identified and the influence of the dynamic characteristics of the resonator on those branches is analysed together with some properties of the band structure. By approximating the ring displacements of the discrete Lagrangian model as a continuum field and through an application of the generalized macro-homogeneity condition, a generalized micropolar equivalent continuum has been derived, together with the overall equation of motion and the constitutive equation given in closed form. The validity limits of the dispersion functions provided by the micropolar model are assessed by a comparison with those obtained by the discrete model.



## 1. Introduction

It is well known that the propagation of elastic waves may be strongly affected by periodic arrangement of scatterers in the material microstructure, in which the density and the elastic constants are periodic function of the position. This has spurred many researches on new materials such as phononic crystals and metamaterials for the control of vibrational waves (see Lu et al., 2009; Pennec et al., 2010; Deymier, 2013; Craster and Guenneau, 2013). In fact, the periodicity of the material microstructure may lead to destructive interferences inducing attenuation of the amplitude of the travelling waves for some bands of frequencies called acoustic wave spectral gap or band gaps.

Lattice materials are phononic crystals whose properties have been extensively studied from the seminal book of Brillouin (1953). Band gaps in periodic lattices were observed by Phani et al. (2006), who analysed the band structure of beam-lattices with ligaments rigidly connected at the nodes and modelled as Timoshenko beams with uniformly distributed mass. This acoustic behaviour may be markedly affected by the lattice topology, as shown by Wang et al. (2015), and is particularly attractive in the case of auxetic lattices (see Prawoto, 2012) because of their dispersive properties (see Krödel et al., 2014). In this class of materials, the chiral lattices have raised the interest of many researchers and several theoretical and experimental studies on their phononic properties have been carried out. A special attention has been devoted to hexachiral lattices, made up of circular rings each of them connected to its neighbours with six ligaments tangent to the ring itself, whose constitutive equation were firstly obtained by Prall and Lakes (1997). Spadoni et al. (2009), carried out a computational investigation on dispersive wave propagation in hexachiral lattices made up of elastic rings and ligaments with distributed mass. The periodic cell was analysed with Bloch boundary conditions for several ratios between the length of the ligaments and the diameter of the rings and band gaps in the frequency spectrum were obtained. A band gap structure for plane tetrachiral lattices was experimentally obtained and numerically simulated by Tee et al. (2010). Based on the Bloch approach incorporating the finite element method, the analysis of the acoustic properties of trichiral lattices developed by Xu et al., 2013, showed the existence of low frequency band-gaps for this lattice topology.

Hexa- and tetra-chiral beam-lattices have also described through equivalent continua mainly based on the micropolar model (see Spadoni and Ruzzene, 2012; Liu et al., 2012; Chen et al., 2014; Bacigalupo and Gambarotta, 2014a; Bacigalupo and De Bellis; 2015). Dispersive functions for homogenized hexachiral and tetrachiral lattices have been obtained by Liu et al. (2012) and

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http://dx.doi.org/10.1016/j.ijsolstr.2016.01.005
0020-7683/

Chen et al. (2014), respectively. However, in the domain of the considered wave numbers, these equivalent micropolar models do not exhibit band gaps.

To obtain low frequency band gaps, the insertion in the microstructure of local resonators generally made of a hard core surrounded by a soft coating has been proved particularly effective. In fact, the locally resonant material may exhibit the emergence of stop bands at frequencies around the natural frequency of the resonator with overall negative mass density and bulk modulus (see for instance Liu et al., 2000; Huang et al., 2009a, 2009b; Lai et al., 2011; Raghavan and Srikantha Phani, 2013; Krushynska et al., 2014). Chiral periodic metamaterials with internal locally resonant structures supporting tunable low-frequency stop bands have been recently proposed by Liu et al. (2011a), Bigoni et al. (2013), and Zhu et al. (2014). In particular, Liu et al. (2011a), have shown that the coupling between the local translational and rotational resonances, stemming from the chiral microstructure, may result in low frequency band gaps and other exotic acoustical effects. Hexachiral beam-lattices integrated with local resonators made up of a softly coated heavy cylinder located inside the rings were analysed numerically by Liu et al. (2011b), for low-frequency wave applications. Through a finite element analysis of the periodic cell with Bloch boundary conditions, the dispersive functions were derived in the reduced Brillouin domain and low-frequency band gaps were obtained. The influence of internal resonators on the band structure of tetra-chiral lattices was analysed by Zhu et al. (2015), through an FE analysis of the periodic cell. These numerical results, corroborated by comparisons with those obtained by experiments, confirmed the internal resonators as effective devices to obtain low frequency band gaps.

The present paper is focused on understanding of the acoustic behaviour of these chiral beam-lattices with two different aims. A first issue concerns the assessment of the influence of the microstructure chirality and of the dynamic characteristics of the resonators on the acoustic behaviour of the lattice, with particular interest in the formation of low-frequency band gaps. With this purpose, a simplified model of hexachiral and tetrachiral lattices (see Fig. 1) has been considered, having a reduced number of degrees of freedom and therefore able to provide some useful analytical results to appreciate the sensitivity of the acoustic band structure on both the lattice chirality and on the properties of the resonator. The ligaments are connected to the rings according to different configurations, ranging from the achiral geometry, with the ligaments normal to the ring, to that of maximum chirality, with the ligaments tangent to the ring. The ligaments are modelled as massless Euler–Bernoulli beams and the rings are assumed rigid and equipped with mass, as well as the cylindrical mass of the resonator. The first assumption relies on the consideration that inertial forces along the ligaments are negligible in the low frequency wave propagation, which is the case of interest of this study, while the second one applies with increasing ring thickness and is corroborated by some numerical simulations in low frequency regime (see Zhu et al. (2015), Fig. 6). A simple Lagrangian model is formulated, which allows the determination of dispersive elastic waves and provides a simple evaluation and a comparison of the effects of the chirality with those of the local resonators.

A further issue concerns the formulation of a homogenized continuum model equivalent to the discrete Lagrangian. The homogenization of beam-lattice models has been tackled by several authors generally referring to homogeneous micropolar models (see for reference Bazant and Christensen, 1972; Noor et al., 1978; Chen et al., 1998; Pradel and Sab, 1998; Forest and Pradel, 2001; Onk, 2002; Ostoja-Starzewski, 2002; Kumar and McDowell, 2004; Gonnella and Ruzzene, 2008a,2008b; Bacigalupo and Gambarotta 2014a,2014b). On the other side, the wave propagation analysis through the dynamic homogenization of beam lattices has been analysed by Suiker et al. (2001); Ostoja-Starzewski (2002); Gonnella and Ruzzene (2008b); Stefanou et al. (2008); Vasiliev et al. (2010). Here, the discrete model above described is homogenized through a generalized energy equivalence criterion, by considering an approximation of the generalized displacement field through a second order Taylor expansion according to an approach proposed by Bazant and Christensen (1972), and applied by Kumar and McDowell (2004) and Liu et al. (2012). The equations of motion thus obtained are those of a generalized micropolar continuum characterized by a generalized displacement field equipped with six degrees of freedom. It may be shown that these equations coincide with those derived by substituting the second order Taylor approximation of the displacement field in the equation of motion of the discrete model.

In order to investigate the influence of chirality on low frequency band gaps, both hexachiral and tetrachiral beam lattices are analysed, respectively, and the dispersion functions of the discrete and of the homogenized model are given, respectively, for several chiral angles measuring the inclination of the ligaments with respect to the line grid joining the centres of the rings. For both the lattices, two acoustic modes and four optical modes are identified and the influence of the dynamic characteristics of the resonator on those branches is analysed together with some properties of the band structure. The validity limits of the micropolar model with respect to the dispersion functions are assessed by comparing the dispersion curves of this model in the irreducible Brillouin domain with those obtained by the discrete model, the latter ones being exact within the assumptions of the proposed simplified model.

## 2. Chiral lattice with local resonators: a simplified model

The beam-lattices shown in Fig. 1 are based on the hexachiral and tetrachiral periodic cells shown in Fig. 2, respectively. Each cell having size $a$ is made up of a ring with mean radius $r$ and $n$ (= 4,6) slender ligaments of length $l$, section width $t$ and unit thickness, rigidly connected to the rings. The inclination of each ligament is denoted by the angle $\beta$ with respect to the lines connecting the centres of the rings. A heavy disk with external radius $R$ shown in Fig. 2 (in dark grey), is located inside the ring through a soft elastic annulus (in yellow). This inclusion plays the role of low-frequency resonator. Increasing the angle $\beta$, a chiral microstructure with auxetic behaviour is obtained up to the condition in which the ligaments are tangent to the ring, when the angle takes the value $\beta_m = \arcsin(\frac{2r}{a})$. This geometry allows considering separately the effects of both the chiral microstructure and the local resonator on the acoustic behaviour of the beam lattice. For $\beta \to 0$ the microstructure is no longer chiral, while for $R \to 0$ the resonator disappears. It follows that the independent geometric parameters of the model are: $a$, $r$, $R$, $t$ and $\beta$. The hexachiral lattice is transversely isotropic while the tetrachiral material belongs to the tetragonal system (see Bacigalupo and Gambarotta, 2014a,2014b).

To obtain a simplified dynamic model (compare with Liu et al., 2011; Zhu et al., 2015) the rings (see Liu et al., 2012) and the disc of the resonators are assumed rigid. The inertia of the elastic soft coating and of the ligaments is ignored, the latter being negligible in the low frequency wave propagation, which is the condition considered in the present study. The Young modulus of the ligaments is denoted by $E_s$, while the rings have mass density $\rho_s$, so that the translational and the rotatory inertia of the rings are $M_1 = 2\pi \rho_s rt$ and $J_1 = M_1 r^2$, respectively. The soft elastic coating inside the resonator has Young's modulus $E_a$ and Poisson's ratio $\nu_a$. The mass density of the internal resonator is denoted by $\rho_a$, so that its translational and rotatory inertia are $M_2 = \pi \rho_a R^2$ and $J_2 = \frac{1}{2} M_2 R^2$, respectively.

The motion of the rigid ring is denoted by the displacement vector $\mathbf{u}$ and the rotation $\phi$, respectively, while the motion of the