International Journal of Solids and Structures 78-79 (2016) 125–130

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Exact theory for a linearly elastic interior beam

Anssi T. Karttunen*, Raimo von Hertzen

Department of Applied Mechanics, School of Engineering, Aalto University, FI-00076 Aalto, Finland

ARTICLE INFO

Article history:
Received 24 November 2014
Revised 7 August 2015
Available online 19 October 2015

Keywords:
Plane beam
Elasticity solution
Airy stress function
Kinematic variables
Clapeyron's theorem
Finite element

ABSTRACT

In this paper, an elasticity solution for a two-dimensional (2D) plane beam is derived and it is shown that the solution provides a complete framework for exact one-dimensional (1D) presentations of plane beams. First, an interior solution representing a general state of any 2D linearly elastic isotropic plane beam under a uniform distributed load is obtained by employing a stress function approach. The solution excludes the end effects of the beam and is valid sufficiently far away from the beam boundaries. Then, three kinematic variables defined at the central axis of the plane beam are formed from the 2D displacement field. Using these central axis variables, the 2D interior elasticity solution is presented in a novel manner in the form of a 1D beam theory. By applying the Clapeyron's theorem, it is shown that the stresses acting as surface tractions on the lateral end surfaces of the interior beam need to be taken into account in all energy-based considerations related to the interior beam. Finally, exact 1D rod and beam finite elements are developed by the aid of the axis variables from the 2D solution.

1. Introduction

Elasticity solutions for plane beams are of fundamental interest in mechanical sciences. An important application of such solutions is the benchmarking of beam theories based on various kinematic assumptions. Two-dimensional (2D) interior elasticity solutions can be easily obtained, for example, for an end-loaded cantilever and a uniformly loaded simply-supported beam by employing the Airy stress function (e.g., Timoshenko and Goodier, 1970).

An interior solution excludes, by virtue of the Saint Venant's principle, the end effects that decay with distance from the ends of a beam. In the calculation of displacements, constraint conditions are applied at the beam supports to prevent it from moving as a rigid body. These constraints for the 2D elasticity solution can be chosen so that they correspond to the boundary conditions of, for example, the Timoshenko beam theory (Timoshenko, 1921). Due to the foregoing, a 2D interior plane stress solution for a plane beam acts as an ideal reference solution for narrow one-dimensional (1D) shear-deformable beam models that do not include end effects.

Many beam and plate theories are based on an assumed displacement field similar to the one first used by Vlasov (1957). These theories are commonly referred to as third-order theories because third-order polynomials are used in the expansion of the displacement components. For surveys on third-order kinematics and plate theories, see the works by Jemielita (1990) and Reddy (1990, 2003). Two examples of third-order beam theories are the Levinson and the Reddy–Bickford beams for which the assumed displacement field is exactly the same (Bickford, 1982; Heyliger and Reddy, 1988; Levinson, 1981; Reddy, 1984). As first shown by Bickford (1982), the Reddy–Bickford beam exhibits a boundary layer character, that is, the decaying end effects are present in the beam. The Reddy–Bickford theory is obtained through an energy-based variational formulation, which results in additional higher-order load resultants in comparison to an interior elasticity solution. If the higher-order load resultants are neglected, the Levinson theory is obtained.

In this study, a general interior elasticity solution is derived for a uniformly loaded linearly elastic homogeneous isotropic 2D plane beam. As the main novelties of the study we find that

- The 2D solution provides the exact third-order kinematics for the beam and can be presented directly in the form of a conventional 1D beam theory.
- By applying the Clapeyron's theorem, it is shown that the stresses acting as surface tractions on the lateral end surfaces of the interior beam are an intrinsic part of all energy-based considerations.
- The 2D solution can be discretized in order to obtain 1D rod and beam finite elements, which provide exact 2D interior displacement and stress distributions.

In more detail, the paper is organized as follows. In the introductory Section 2, a polynomial Airy stress function is used to derive the interior stress field for a 2D plane beam under a uniform distributed load. The strains are calculated from the stresses according to the plane stress condition and the displacements are integrated

* Corresponding author. Tel.: +358405272713.
E-mail addresses: anssi.karttunen@aalto.fi (A.T. Karttunen), raimo.von.hertzen@aalto.fi (R. von Hertzen).

http://dx.doi.org/10.1016/j.ijsolstr.2015.09.010
0020-7683/

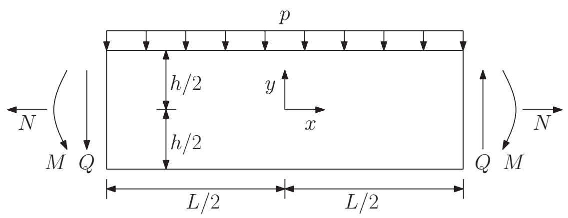

Fig. 1. 2D homogeneous isotropic plane beam with a rectangular cross-section under a uniform pressure. The load resultants act at an arbitrary cross-section of the beam.

from the strains. In Section 3, three kinematic variables defined at the central axis of the plane beam are formed from the 2D interior displacement field. Using these new central axis variables, 1D beam equations are developed. The total potential energy of the interior beam and Clapeyron's theorem are considered. Finally, exact 1D interior rod and flexural beam finite elements are developed from the 2D interior elasticity solution. Conclusions are presented in Section 4.

2. Stress function solution for a plane beam

2.1. Plane beam problem and Airy stress function

Fig. 1 presents a 2D homogeneous isotropic plane beam under a uniform pressure p. The beam has a rectangular cross-section of constant thickness t and the length and height of the beam are L and h, respectively. The load resultants N, M and Q stand for the axial force, bending moment and shear force, respectively. These cross-sectional load resultants are calculated from the equations

$$N(x) = t\int_{-h/2}^{h/2} \sigma_x(x,y)dy, \quad M(x) = t\int_{-h/2}^{h/2} \sigma_x(x,y)ydy,$$
$$Q(x) = t\int_{-h/2}^{h/2} \tau_{xy}(x,y)dy, \tag{1}$$

which can be used to impose the force and moment boundary conditions at $x = \pm L/2$. The boundary conditions on the upper and lower surfaces of the beam are

$$\sigma_y(x, h/2) = -p, \quad \sigma_y(x, -h/2) = 0, \quad \tau_{xy}(x, \pm h/2) = 0. \tag{2}$$

Note that the boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces, whereas at the beam ends the tractions are not specified at each point but only through the load resultants and, thus, the boundary conditions are imposed only in a *weak* sense (Barber, 2010). In the case of Fig. 1, the replacement of the true boundary conditions at the beam ends by the statically equivalent weak boundary conditions (load resultants) implies that the exponentially decaying end effects of the plane beam are neglected by virtue of the Saint Venant's principle and only the interior solution of the beam is under consideration. The interior solution represents essentially a beam section which has been cut off from a complete beam far enough from the real lateral boundaries at which the true boundary conditions could be set. The stresses of the plane beam are obtained from the equations

$$\sigma_x = \frac{\partial^2\Psi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Psi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Psi}{\partial x\partial y}, \tag{3}$$

where $\Psi(x, y)$ is the Airy stress function. Eqs. (3) satisfy the two-dimensional equilibrium equations. To ensure compatibility, it is required that the stress function satisfies the biharmonic equation (Barber, 2010)

$$\frac{\partial^4\Psi}{\partial x^4} + 2\frac{\partial^4\Psi}{\partial x^2\partial y^2} + \frac{\partial^4\Psi}{\partial y^4} = 0. \tag{4}$$

The solution to the plane beam problem is obtained by finding a solution of Eq. (4) that satisfies the stress boundary conditions (2) of the beam.

2.2. Interior stress field of a plane beam

By adapting a general solution procedure outlined by Barber (2010, chap. 5), we find that the polynomial stress function for the interior problem of any plane beam under a uniform pressure p (see Fig. 1) is

$$\Psi(x,y) = c_1y^2 + c_2y^3 + c_3xy\left(1 - \frac{4y^2}{3h^2}\right) - \frac{q}{240I}\left[5h^3x^2 + 15h^2x^2y + 4y^3\left(y^2 - 5x^2\right)\right], \tag{5}$$

where $q = pt$ is the uniform load, $I = th^3/12$ is the second moment of the cross-sectional area and c_1, c_2 and c_3 are to be solved by the aid of Eqs. (1). The stresses calculated from Eqs. (3) are

$$\sigma_x = 2c_1 + 6c_2y - \frac{8c_3xy}{h^2} + \frac{q(3x^2y - 2y^3)}{6I}, \tag{6}$$

$$\sigma_y = -\frac{q}{24I}(h^3 + 3h^2y - 4y^3), \tag{7}$$

$$\tau_{xy} = c_3\left(\frac{4y^2}{h^2} - 1\right) + \frac{qx}{2I}\left(\frac{h^2}{4} - y^2\right). \tag{8}$$

Note that the above interior stress distributions are universal, that is, they are valid for any plane beam under a uniform load since they are not associated with any particular constraint conditions at the beam ends. Using Eqs. (6) and (8), the load resultants calculated from Eqs. (1) are

$$N = 2Ac_1, \quad M = 6Ic_2 - \frac{2}{3}Ac_3x + \frac{q}{2}\left(x^2 - \frac{h^2}{10}\right), \quad Q = qx - \frac{2}{3}Ac_3, \tag{9}$$

where $A = ht$ is the area of the cross-section. As a first step towards presenting the solution in the form of a 1D beam theory, it can be easily verified that the following global equilibrium equations, which can also be obtained by integrating the 2D stress equilibrium equations, hold for the load resultants (9)

$$\frac{\partial N}{\partial x} = 0, \quad \frac{\partial M}{\partial x} = Q, \quad \frac{\partial Q}{\partial x} = q. \tag{10}$$

We note that Schneider and Kienzler (2015) arrived at the same equilibrium equations (10) through their recent exact 3D representation of linear elasticity. When c_1, c_2 and c_3 are solved from Eqs. (9) and substituted into Eqs. (6) and (8), we obtain

$$\sigma_x = \frac{N}{A} + \frac{My}{I} + \frac{3qy}{5A} - \frac{qy^3}{3I}, \tag{11}$$

$$\tau_{xy} = \frac{Q}{8I}(h^2 - 4y^2). \tag{12}$$

The stress distribution of Eq. (11) has been called by Rehfield and Murthy (1982) the refined (nonclassical) axial stress distribution in the context of their beam theory. More complicated distributed loads lead to different additional load terms in the stresses. By setting $q = 0$, Eqs. (11) and (12) give the stress distribution of the classical Euler–Bernoulli beam.

2.3. Example – Simply-supported beam

As an example, let us consider a simply-supported beam under a constant uniform load q. In a setting according to Fig. 1, the axial force, bending moment and shear force along the beam are given by

$$N(x) = 0, \quad M(x) = q(x^2/2 - L^2/8), \quad Q(x) = qx, \tag{13}$$

respectively. We find that the interior stress state in the beam calculated from Eqs. (7) and (11)–(13) is the same as the one found in any