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Bifurcation of a dielectric elastomer balloon under pressurized inflation and electric actuation

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ABSTRACT

It is previously known that under inflation alone a spherical rubber membrane balloon may bifurcate into a pear shape when the tension in the membrane reaches a maximum, but the existence of such a maximum depends on the material model used: the maximum exists for the Ogden model, but does not exist for the neo-Hookean, Mooney–Rivlin or Gent model. This paper discusses how such a situation is changed when a pressurized dielectric elastomer balloon is subjected to additional electric actuation. A similar bifurcation condition is first deduced and then verified numerically by computing the bifurcated solutions explicitly. It is shown that when the material is an *ideal* dielectric elastomer, bifurcation into a pear shape is possible for all material models, and similar results are obtained when a typical non-ideal dielectric elastomer is considered. It is further shown that whenever a pear-shaped configuration is possible it has lower total energy than the co-existing spherical configuration.

1. Introduction

Dielectric elastomers are now widely recognized as a high-tech engineering material that has a variety of applications, ranging from robotics where it is used as artificial muscles, to energy harvesting where it is used to convert mechanical energy into electricity. They have received a lot of attention since they were first reported (Fox and Goulbourne, 2008; Kofod, 2008; Patrick et al., 2007; Pelrine and Prahlad, 2008). A key question that is addressed by many recent studies is their shape bifurcation and its effect on the performance and reliability of structures/devices made from such soft materials; see, e.g., Plante and Dubowsky (2006), Bertoldi and Gei (2011), De Tommasi et al. (2013) and Dorfmann and Ogden (2014a), and the references therein.

There are many types of configurations for the generators and actuators made from dielectric elastomers, and one of them is the spherical balloon shape (Artusi et al., 2011; Soleimani and Menon, 2010). A procedure was presented by Ahmadi et al. (2013) for fabricating and testing a seamless spherical dielectric elastomer balloon. Various aspects of the uniform inflation problem, such as the so-called limiting-point instability, have been examined by Mockensturm and Goulbourne (2006), Zhu et al. (2010), He et al. (2011), Rudykh et al. (2012), Keplinger et al. (2012), and Dorfmann and Ogden (2014b). York et al. (2010) and De Tommasi et al. (2014) studied the hysteresis effects commonly exhibited in such structures.

Since a spherical balloon is an important configuration in the application of dielectric elastomers, it is also of interest to understand whether shape bifurcation will take place when it is subjected to the combined action of internal inflation and a voltage. When a membrane balloon is under internal inflation alone, it is well known that the spherical shape may bifurcate into a pear shape when the internal volume reaches a first critical value, and then return to a spherical shape at a second, higher critical value (Ericksen, 1998; Feodosev, 1968; Haughton and Ogden, 1978). Chen and Healey (1991) showed that the pear-shaped configuration must necessarily have lower energy than the co-existing spherical configuration, and they also derived some sufficient conditions under which the above bifurcation behavior actually occurs for a general material model. Fu and Xie (2014) analyzed the stability of the pear-shaped configuration itself with respect to further axi-symmetric perturbations, and showed that it is stable under mass or volume control but unstable under pressure control. The well-known bifurcation condition in the purely mechanical case was originally derived from the incremental theory of nonlinear elasticity, but it was shown in Fu and Xie (2014) that if attention is focused on axi-symmetric bifurcation modes then bifurcation can be detected by a simple shooting procedure based on the original governing equations. It is this latter method that will

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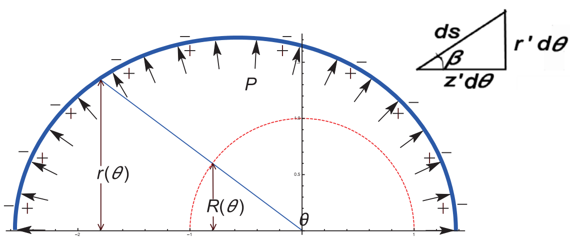

Fig. 1. Undeformed and deformed configurations of a dielectric elastomer balloon under both pressurized inflation and electric actuation.

be employed in the present paper. Our main objective is to understand how adding electric actuation affects the appearance of the pear-shaped configurations. We first extend the bifurcation condition for the purely mechanical case in a straightforward manner, and then verify that the extended bifurcation condition is indeed valid by using the above-mentioned shooting procedure.

The remainder of this paper is organized as follows. Section 2 presents the governing equations for the axi-symmetric deformations of a dielectric elastomer spherical balloon. These equations are then solved in Section 3 to find pear-shaped configurations that may bifurcate from the spherical configuration, guided by the bifurcation condition that was extended from its counterpart in the purely mechanical case. The total energy is computed to demonstrate that whenever a pear-shaped configuration can exist it is preferred to the co-existing spherical configuration. We conclude the paper in Section 4 with a discussion of how electric actuation based on different material models affects the appearance of the pear-shaped configurations.

2. Governing equations

Fig. 1 shows the upper half of a spherical balloon that is made of a dielectric elastomer material and is subjected to both internal inflation and electric actuation of the compliant electrodes attached to the inner and outer surfaces. Without loss of generality, the initial radius is assumed to be unity (which is equivalent to using the initial radius as the unit of length), and so the undeformed configuration Ω is described by

$$R(\theta) = \sin\theta, \qquad Z(\theta) = 1 - \cos\theta, \qquad 0 \le \theta \le \pi$$

in terms of cylindrical polar coordinates. We focus on axi-symmetric deformations (spherical or pear-shaped) described by

$$r = r(\theta), \qquad z = z(\theta), \qquad 0 \le \theta \le \pi,$$

where (r, z) are cylindrical polar coordinates in the current configuration (note that θ is not a polar coordinate; it is simply a parameter). The principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \qquad \lambda_2 = \sqrt{r'^2 + z'^2}, \qquad \lambda_3 = \frac{h}{H},$$

where a prime denotes differentiation with respect to θ, and H and h are the thicknesses in the reference and deformed configurations, respectively.

According to Dorfmann and Ogden (2005), the total energy density function $\hat{W}$ of an incompressible dielectric elastomer can be assumed to be a function of the five invariants I_1, I_2, I_4, I_5, I_6 defined by

$$I_1 = \operatorname{tr} C, \quad I_2 = \operatorname{tr} C^{-1}, \quad I_4 = \boldsymbol{D}_l \cdot \boldsymbol{D}_l, \\ I_5 = \boldsymbol{D}_l \cdot C\boldsymbol{D}_l, \quad I_6 = \boldsymbol{D}_l \cdot C^2\boldsymbol{D}_l, \tag{1}$$

where $C\ (= F^T F)$ is the right Cauchy–Green deformation tensor, F is the deformation gradient, and $\boldsymbol{D}_l$ is the nominal electric displacement which is related to the true electric displacement $\boldsymbol{D}$ by $\boldsymbol{D}_l = F^{-1}\boldsymbol{D}$.

Following Dorfmann and Ogden (2014a), we shall consider the following simplest constitutive law that accounts for electro-elastic coupling:

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2) + \frac{1}{2}\epsilon_0^{-1}(\xi I_4 + \eta I_5), \tag{2}$$

where $W(\lambda_1, \lambda_2)$ is the strain–energy density function of the elastomer per unit volume in the reference configuration, ϵ_0 is the vacuum permittivity, and ξ, η are two dimensionless material constants that characterize electroelastic coupling. The second term on the right hand side of (2) denotes the free energy associated with polarization induced by the voltage. The above model reduces to that of an ideal dielectric elastomer when $\xi = 0$ and η is equal to ϵ_0 divided by the permittivity; see, e.g., Zhao and Suo (2007).

The electric field $\boldsymbol{E}$ is computed from $\boldsymbol{E} = F^{-T}\partial\hat{W}/\partial\boldsymbol{D}_l$, and is given by

$$\boldsymbol{E} = \epsilon_0^{-1}(\xi B^{-1}\boldsymbol{D} + \eta\boldsymbol{D}), \tag{3}$$

where $B = FF^T$. For the problem under consideration where a voltage ϕ is specified, we have

$$\boldsymbol{E} = \frac{\phi}{h}\boldsymbol{e}_3 = \frac{\phi}{H}\lambda_1\lambda_2\boldsymbol{e}_3,$$

where $\boldsymbol{e}_3$ denotes the unit vector normal to the membrane surface. It then follows that:

$$\boldsymbol{D} = \epsilon\kappa(\lambda_1, \lambda_2)\boldsymbol{E}, \quad I_4 = \mu\Phi\lambda_1^4\lambda_2^4\epsilon\kappa^2(\lambda_1, \lambda_2), \quad I_5 = (\lambda_1\lambda_2)^{-2}I_4, \tag{4}$$

where

$$\epsilon = \epsilon_0/(\xi + \eta), \quad \Phi = \epsilon\phi^2/\left(\mu H^2\right), \\ \kappa(\lambda_1, \lambda_2) = (\xi + \eta)/\left(\xi\lambda_1^2\lambda_2^2 + \eta\right). \tag{5}$$

In the above expressions, the constants μ and ϵ denote the shear modulus and permittivity when there is no deformation, whereas the product $\epsilon\kappa(\lambda_1, \lambda_2)$ may be interpreted as the deformation-dependent permittivity corresponding to the simple model (2). It is then appropriate to impose the inequalities $0 < \xi + \eta \le 1$. We observe that in the subsequent analysis the ground state permittivity ϵ will only appear through the non-dimensional parameter Φ, and ξ and η will always appear in the form ξ/η.

With the application of both a voltage ϕ and inner pressure P, the total free energy in the system takes the form

$$E = \int_\Omega \hat{W}(\lambda_1, \lambda_2)dV - Pv - \phi Q, \tag{6}$$

where v is the volume enclosed by the inner surface of the deformed balloon, and Q is the charge accumulated on each electrode. The Q and v may be calculated using the formulae

$$Q = \int_\Gamma \frac{\epsilon\kappa(\lambda_1, \lambda_2)\phi}{h}da = \int_0^\pi \frac{\epsilon\kappa(\lambda_1, \lambda_2)\phi}{h} \cdot 2\pi r\lambda_2 d\theta, \\ v = \int_0^\pi \pi r^2 z' d\theta,$$

where Γ denotes the current configuration of the inner surface. With the use of these expressions and introduction of a new *effective strain-energy function* $\tilde{W}$ defined by

$$\tilde{W} \equiv \mu^{-1}\hat{W}(\lambda_1, \lambda_2) - \Phi\kappa(\lambda_1, \lambda_2)\lambda_1^2\lambda_2^2, \tag{7}$$

Eq. (6) can be simplified to

$$E/(2\pi H\mu) = \int_0^\pi \mathcal{L}(\boldsymbol{u}, \boldsymbol{u}')d\theta, \tag{8}$$

where $\boldsymbol{u} = (r(\theta), z(\theta))$, and

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{u}') = \tilde{W}(\lambda_1, \lambda_2)\sin\theta - \frac{1}{2}\bar{P}r^2 z', \quad \bar{P} = P/(\mu H). \tag{9}$$