International Journal of Solids and Structures 73–74 (2015) 67–77

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

# A sensitive interval of imperfect interface parameters based on the analysis of general solution for anisotropic matrix containing an elliptic inhomogeneity

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ARTICLE INFO

*Article history:*
Received 12 September 2014
Received in revised form 25 June 2015
Available online 23 July 2015

*Keywords:*
Anisotropic materials
Elliptic inhomogeneity
Imperfect interface
Plane deformation
Complex function method

ABSTRACT

A general solution is proposed to solve the plane problem for infinite anisotropic medium containing an elliptic inhomogeneity with imperfect interface. The imperfect interface is usually described as a spring model with vanishing thickness based on the assumption that the tractions are continuous but the normal or tangential displacements are discontinuous due to a jump at the interface. By means of the series expansion of the complex stress functions and the factor functions for the imperfect elliptic interface, a general procedure to determine the coefficients in the series is illustrated and the convergent solutions are obtained by truncating finite number of terms in the series. The present solutions are verified with available analytical results for the cases of perfect interface and debonded interface (or hole). The patterns of the stresses in the anisotropic medium (or matrix) and inhomogeneity due to the eigenstrains and far-filed stresses are presented, respectively. A sensitive interval of interface parameters is suggested, in which the influence of the change of interface parameters on the stress field is very obvious. Monotonic and non-monotonic change of peak stresses on a subset of interface parameters is also discussed. The method and the procedure proposed in this work can be used in the analysis of strength and failure of anisotropic materials containing an elliptic inhomogeneity with imperfect interface.



## 1. Introduction

After Eshelby (1957, 1959, 1961) studied an inclusion problem and provided a general solutions for the elastic field of isotropic materials in the case of perfect interface between matrix and inclusions in earlier years, many techniques such as Green function method, Integral transform method and Stroh method have been developed by researchers to solve the problems for materials containing inhomogeneity (or inclusion). Numerous problems of two-phase inhomogeneity/inclusion bonded within isotropic materials have been investigated in the case of uniform or polynomial form of eigenstrains in the inhomogeneity. Recently, the plane problem for the polynomial eigenstrains in the elliptic inhomogeneity bonded within infinite anisotropic materials has been studied by Nie et al. (2007, 2009, 2014), and some analytic results for orthogonal anisotropic matrix containing an elliptic inhomogeneity with perfect interface was obtained based on the complex function method and the principle of minimum strain energy.

For the imperfect interface, many models were presented and studied by a lot of researchers such as Hashin (1991, 2002) and Hashin and Monteiro (2002) and Benveniste and Miloh (2001). One of them is the spring model with continuous condition for the tractions and discontinuous condition for a jump in the normal or tangential displacements, in which the jumps are linearly proportional to their respective traction at the interface. Its influence on the properties of materials has been receiving a lot of authors' attention. However, most of the analytic results are limited to special problems, such as isotropic matrix containing a circular inhomogeneity (or inclusion) with sliding interface (Ru, 1998) and dislocation interface (Kattis and Providas, 1998).

Many researchers investigated the influence of changing parameters of the imperfect interface on properties of materials by using the semi-analytical analysis (Kushch and Chernobai, 2014; Wang et al., 2014) and the numerical simulations (Würkner et al., 2013, 2014; Nairn, 2007). Numerous studies for the influence on stresses in the constituents of materials were also carried out by many researchers, for example, Shen et al. (2000, 2001a,b, 2005), Wang et al. (2005), Wang (2006), Ting and Schiavone (2010), Nie and Huang (2009) as well as Chan et al.

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http://dx.doi.org/10.1016/j.ijsolstr.2015.07.009
0020-7683/

(2010), using the plane and anti-plane elasticity-based semi-analytical and numerical analysis. Because a very simple change of imperfect parameters is used in the analysis of the results and the expansion terms in truncating the series for numerical calculation is few, neither the sensitive interval nor is the mechanism of the stress change very clear.

The general solution for anisotropic plane with perfectly bonded elliptic inhomogeneity was obtained using Lekhnitskii's theory. The problem of imperfect bonded elliptic inhomogeneity is considered in this work based on the theory. The interaction on imperfect interface is simulated using the spring model with a vanishing thickness, in which the continuous condition is assumed for the traction and the discontinuous condition is assumed for a jump in normal and tangential displacements. Solution of the plane problem for the anisotropic medium containing an elliptic inhomogeneity is presented for the imperfect interface. By means of the complex series expansion of the stress functions and factor functions for the elliptic imperfect interface, a general procedure for determining the coefficients in the series is also presented. The series expansion of the factor functions ($H^{(l)},\ l=1,\ 2$) could be used to solve some problems of inhomogeneous interface parameters for the spring model in the future research. Introducing the governing equations for describing the stress fields of the matrix and the inhomogeneity, resulting sets of algebraic equations for the unknown coefficients are solved by truncating finite number of terms in the series. Convergence of the solutions is examined by truncating various terms in the series, and the present solutions are verified with available analytical results for the cases of perfect interface and debonded interface (or hole). Detailed analysis is given for the influence of the interface parameters on the whole stress field both due to eigenstrains and far-field stresses, and a sensitive interval for the influence of interface parameters on macro stress field is suggested for the analysis of material property. Finally, the method and procedure proposed in this paper can be used in the analysis of strength and failure of anisotropic materials containing an elliptic inhomogeneity with imperfect interface.

## 2. Fundamental equations for plane problems

For the plane problem, two equilibrium equations without considering body forces will be satisfied by choosing a representation

$$\sigma_x=\frac{\partial^2F}{\partial y^2},\quad \sigma_y=\frac{\partial^2F}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2F}{\partial x\,\partial y},\tag{2.1}$$

where $\sigma_x,\ \sigma_y$ and $\tau_{xy}$ are the components of stress, $F(x,y)$ is an arbitrary form called the Airy stress function.

For plane stress problem, the constitutive relations may be expressed as

$$\left.\begin{aligned}\varepsilon_x&=a_{11}\sigma_x+a_{12}\sigma_y+a_{16}\tau_{xy}\\ \varepsilon_y&=a_{12}\sigma_x+a_{22}\sigma_y+a_{26}\tau_{xy}\\ \gamma_{xy}&=a_{16}\sigma_x+a_{26}\sigma_y+a_{66}\tau_{xy},\end{aligned}\right\}\tag{2.2}$$

where $\varepsilon_x,\ \varepsilon_y$ and $\gamma_{xy}$ are strain components. Substituting Eq. (2.2) together with Eq. (2.1) into the compatibility equation, a solution to the compatibility equation in terms of the stress function has the form of

$$F(x,y)=F_1(x+\mu_1y)+F_2(x+\mu_2y)+F_3(x+\mu_3y)+F_4(x+\mu_4y),\tag{2.3}$$

where $\mu_i,\ i=1,\ldots,4$, are four roots of the resulting characteristic equation

$$a_{11}\mu^4-2a_{16}\mu^3+(2a_{12}+a_{66})\mu^2-2a_{26}\mu+a_{22}=0.\tag{2.4}$$

For ideal elastic materials, the four roots correspond to two pairs of complex conjugates such that $\mu_k=\alpha_k+i\beta_k$, ($\beta_k>0,\ k=1,\ 2$) where $\alpha_k$ and $\beta_k$ are real and imaginary parts respectively, and $\mu_3=\bar{\mu}_1,\ \mu_4=\bar{\mu}_2$. The roots ($\mu_k,\ k=1,\ 2$) are two basic complex parameters characterizing the degree of anisotropy. Eq. (2.3) can thus be expressed as

$$F=2\mathrm{Re}\sum_{k=1}^{2}F_k(z_k),\tag{2.5}$$

where

$$z_k=x+\mu_ky,\quad (k=1,2),\tag{2.6}$$

indicating two physical complex planes for anisotropic materials. Introducing two generalized stress functions such that

$$\varphi_k(z_k)=\frac{dF_k}{dz_k},\quad \varphi_k'(z_k)=\frac{d\varphi_k}{dz_k},\quad (k=1,2),\tag{2.7}$$

then

$$\frac{\partial F}{\partial x}=2\mathrm{Re}\sum_{k=1}^{2}\varphi_k(z_k),\quad \frac{\partial F}{\partial y}=2\mathrm{Re}\sum_{k=1}^{2}\mu_k\varphi_k(z_k).\tag{2.8}$$

The stress and displacement components can be expressed as

$$\begin{aligned}\sigma_x(x,y)&=2\mathrm{Re}\sum_{k=1}^{2}\mu_k^2\varphi_k'(z_k),\\ \sigma_y(x,y)&=2\mathrm{Re}\sum_{k=1}^{2}\varphi_k'(z_k),\\ \tau_{xy}(x,y)&=-2\mathrm{Re}\sum_{k=1}^{2}\mu_k\varphi_k'(z_k),\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}u(x,y)&=2\mathrm{Re}\sum_{k=1}^{2}p_k\varphi_k(z_k),\\ v(x,y)&=2\mathrm{Re}\sum_{k=1}^{2}q_k\varphi_k(z_k),\end{aligned}\tag{2.10}$$

respectively, where

$$\begin{aligned}&p_1=a_{11}\mu_1^2+a_{12}-a_{16}\mu_1,\quad p_2=a_{11}\mu_2^2+a_{12}-a_{16}\mu_2,\\ &q_1=a_{12}\mu_1+\frac{a_{22}}{\mu_1}-a_{26},\quad q_2=a_{12}\mu_2+\frac{a_{22}}{\mu_2}-a_{26},\end{aligned}\tag{2.11}$$

in which $p_1,p_2,q_1,q_2$ are four complex coefficients.

## 3. Complex stress functions for infinite matrix with an elliptic inhomogeneity

Suppose homogeneous, linear elastic and infinite anisotropic medium (matrix) contains an elliptic inhomogeneity (or inclusion) in plane elasticity, as shown in Fig. 1. Semi-major and semi-minor axis of the elliptic inhomogeneity are denoted by $a$ and $b$ in the $x$ and $y$ directions, respectively. The interface between the matrix and inhomogeneity is denoted by $\Gamma$. $\varepsilon_x^*,\ \varepsilon_y^*$ and $\gamma_{xy}^*$ are the eigenstrains components in the inhomogeneity, and $\sigma_x^\infty,\ \sigma_y^\infty$ and $\tau_{xy}^\infty$ are the remote uniform loadings in the matrix.

First, a complex plane ($z$-plane) with the real $x$-axis and the orthogonal imaginary $y$-axis is established for the system of the matrix and inhomogeneity shown in Fig. 1. Consider the transformation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the complex $z$-plane for the inhomogeneity into the $z_k$-plane with $Z_k=x+\mu_ky,\ k=1,\ 2$, as shown in Fig. 2(a). The region remains elliptic during the transformation and becomes