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A computational model for large deformations of composites with a 2D soft matrix and 1D anticracks

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A B S T R A C T

Anticracks (also known as rigid line inclusions) occur frequently in a variety of natural and engineered composites as very stiff and extremely sharp (almost zero-thickness) fibers or lamellae embedded in a softer matrix.

In the linear elastic regime, similarly to cracks, anticracks generate a singularity in the stress distribution around the tip. Because of this similarity, existing analytical techniques and solutions (for simple cases) can be easily translated to anticracks. However, despite their importance in many biological and engineering composites, there has been surprisingly little development of numerical methods that would account simultaneously for the presence of multiple fibers or lamellae, arbitrary loadings and nonlinear behavior of the matrix.

This paper presents the first numerical approach for rigid line inclusions, based on a meshfree scheme recently developed for multiple crack growth in elastic media. The inclusion of zero thickness is created as a crack, and a rigid motion (rotation and translation) is enforced at the anticrack faces. The equations of motion are solved according to a Total Lagrangian framework, and the matrix supposed hyperelastic.

Contrarily to available analytical solutions, the degrees of freedom of the rigid motion are determined *a posteriori* as a consequence of the (discretized) elastic equilibrium, expressed in a variational approach.

Results show that the proposed approach match well the analytical solutions and provides accurate stress intensity factors (SIFs) for relatively little computational cost. Moreover, the method can reproduce some peculiar features of the anticracks: unlike cracks, singularities also appear under compressive and parallel loads; moreover, for a certain combination of biaxial load, stress concentrations disappear.

Finally, the paper presents examples drawn from biological and engineering composites: the reorientation of one or more fibers under large strains, resulting in a smart stiffening and strengthening mechanism. Reorienting towards the direction of applied load has structural importance since reinforcements can have the most effectiveness in withstanding loads. If the matrix is compliant, the reorientation is eased.

1. Introduction

A rigid line inclusion (RLI) is a mathematical abstraction of an extremely thin stiff inclusion dispersed within a matrix. The definition assumes the inclusion as infinitely rigid and *zero-thickness*. Kinematically, this model consists in a surface of discontinuity (a crack) where a rigid motion is imposed on all the material points belonging to the upper and lower faces of the inclusion. For this reason, some authors (Hurtado et al., 1996) refer to this model as an *anticrack*. However, in geology this terminology indicates something different: a classical Mode I crack displacement solution with a reversed sign (Fletcher and Pollard, 1981), which in classical fracture mechanics means a violation of the non-penetrability of the crack faces. However, the justification is the dissolution and removal of material when the anticrack surfaces move toward each other, which is useful to explain triggering mechanism for snow slab avalanches (Heierli et al., 2008) or shallow earthquakes (Green et al., 1990; Burnley and Green, 1989). In RLI instead, the impenetrability is automatically imposed by a rigid motion common to both faces.

RLIs are useful to model the effects on the matrix of thin reinforcements in form of fibers, platelets, needles or rods of

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characteristic sizes much smaller than that of the embedding matrix. These reinforcements appear in many biological systems and engineered nanocomposites. For example, in biological systems (Pingle et al., 2008) like bones, teeth or nacre, the reinforcement is usually in mineralized crystal form arranged in a *staggered* disposition within a protein matrix. In calcified tissues, (Landis, 1995), these fibers influence their strength, and the overall effect is a tough nanocomposite (Ji and Gao, 2004; Pugno, 2006) produced from very *poor* materials (Fratzl and Guille, 2011). In engineered nanocomposites, RLIs appear as needle-like reinforcements (Bilotti et al., 2008, 2009, 2010), nanowhiskers (Eichhorn et al., 2010), nanoplatelets (Porwal et al., 2013b,a,c) and carbon nanotubes (Nishimura and Liu, 2004).

Many theoretical papers are available in the literature for the RLI problem, often encountered with different terminology, such as *line stiffener* or *anticrack*, owed to its resemblance with a crack. Most likely, this abundance is due to the application to RLI of already well-known techniques at that time for 2D problems: for instance, the Mushkelishvili solutions in terms of complex variable, and the Wiener–Hopf technique, previously applied for crack problems (Muskhelishvili, 1953).

Probably the first paper on RLI appeared in 1973 (Atkinson, 1973), with the term *ribbon* instead of *rigid line inclusion*. The scope of this paper was to study the response of a metallic strain measuring device in a rubber matrix. This paper presented firstly the solution for stresses in an elastic linear matrix due to a single isolated rigid *ribbon*, and secondly the solution for the *elastic ribbon*. The crack analogy is then exploited to obtain the solution for two collinear rigid inclusions, and finally, the interaction of a RLI with a free boundary. Later, Brussat and Westmann (1975) proved the correspondence between the *Westergaard* stress function for cracks and a stress function for RLI, and subsequently, the relation between their *stress intensity factors* (SIFs). Hasebe et al. (1984) instead proposed a rational mapping function (again taken from the elasticity of cracks) to analyze the stress state near a the tip of a crack initiated from the tip of a RLI. Wang et al. (1985) obtained the asymptotic expansion near the tip of a RLI (reported in Section 2.2 of this paper) for both stress and strain fields. Chen (1986) and later Stagni (1989) proved the path-independence of the *J-integral* around the tip of a RLI, and found that the *J-integral* for an anticrack is negative, rather than positive like in cracks. Dundurs and Markenscoff (1989) and Ballarini (1987) reported a *full-field* solution for the stresses in the matrix due to a RLI, respectively using a *weight function technique* and an *integral equation approach*, and later for a RLI at the interface of two dissimilar materials (Ballarini, 1990). Hurtado et al. (1996) introduced the term *anticrack* for RLI and *quasicracks* for elastic line inclusion: they obtained similar solutions to Atkinson (1973) starting from the Eshelby's ellipsoidal *equivalent inclusion*, for the limit to zero of the ratio between the axes.

Despite the great amount of theoretical work produced over the years, there was no attention to investigate experimentally the stress distribution near a line stiffener, until 2008, when Dal Corso et al. (2008) and Bigoni et al. (2008), and later Dal Corso and Bigoni (2009) and Noselli et al. (2010) interestingly disclosed, with photo-elasticity, the *full-field* stress state of an extremely thin and stiff inclusion made of steel embedded in a transparent epoxy matrix. They validated with their experiments some intriguing aspects of the RLI problem, already known from the analytical solutions: for instance, the appearance of a *square root* singularity also for tensile loading parallel to the stiffener.

With the field of nanocomposites in rapid growth, it becomes of paramount importance to develop numerical methods that implement RLI models that could be used by materials scientists and engineers to investigate the toughness properties of both natural and man-made composites, or to imitate artificially the hierarchical structures present in nature. This topic seems to have been overlooked by researchers in numerical methods, with almost absent literature in this field. It is worth to acknowledge the significant contributions of Radtke et al. (2010, 2011) where they employ a Partition of Unity Finite Element Method (PUFEM) to introduce short thin fibers in a cementitious matrix as a *tunneling crack* with a finite very short thickness, not zero. The *tunnel* is introduced as a two-dimensional *Heaviside* enrichment (1 inside the fiber, 0 otherwise) over the span of the fiber. Instead, we introduce an *exactly zero* thickness. Moreover, in these works it is not reported any connections with a negative *J*-integral, nor comparisons with existing analytical solutions, whereas instead we make use of the relation in Chen (1986) and a numerically computed *J*-integral to validate our results in terms of stress intensity factors.

Exploiting the strong relation with cracks, we used an idea recently developed (Barbieri et al., 2012; Barbieri and Petrinic, 2013b,a) for fracture in a meshfree context: the aim is to create a crack where the RLI is positioned, and then impose a rigid motion at the (anti) cracks surfaces. The orientation of the inclusion can be arbitrary inside the matrix, without restrictions imposed by the underlying discretization of the matrix.

The structure of the paper is the following: Section 2 summarizes the analytical solutions available in the literature, alongside with the formulas for the extraction of the SIFs; Section 3 describes the governing equations in strong and weak form and the ones arising from their discretization; Section 4 presents the examples for the validation of the method, comparison with analytical solutions (full field and SIFs) and reorientation of fibers under a tensile loading; finally, in Section 5 conclusions are drawn.

2. Analytical solutions, J-integral and stress intensity factors

2.1. Analytical solution

Atkinson (1973) derived an analytical solution for an horizontal rigid line inclusion problem in an infinite isotropic elastic matrix under uniform remote biaxial loading σ_x^∞ and σ_y^∞. In the following, the orthogonal reference has axis x aligned with the inclusion with the origin in its middle point. The rigid line inclusion has length $2a$.

Under uniform biaxial tension, and without the inclusion, the matrix strains uniformly, with a displacements field given by

$$u_0(x,y) = \frac{x}{8\mu}\left((\kappa+1)\sigma_x^\infty + (\kappa-3)\sigma_y^\infty\right) \tag{1}$$

$$v_0(x,y) = \frac{y}{8\mu}\left((\kappa-3)\sigma_x^\infty + (\kappa+1)\sigma_y^\infty\right) \tag{2}$$

where κ is

$$\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases} \tag{3}$$

and μ is the *shear modulus* and ν is the Poisson ratio. The component ϵ_{x0} of the strain tensor given by

$$\epsilon_{x0}(x,y) = \frac{1}{8\mu}\left((\kappa+1)\sigma_x^\infty + (\kappa-3)\sigma_y^\infty\right) \tag{4}$$

The line inclusion can only move rigidly. Hence, with the rigid line inclusion now inserted in the matrix, and for the symmetry of the problem, the motion is only translational in the horizontal direction and with no rotation. For the compatibility of the displacements, this translation must be equal to the displacement u_0 (1) at its tips ($x = \pm a,\ y = 0$). In deriving the analytical solution, Atkinson, 1973 conveniently subtracted out the uniform strain of the matrix to obtain zero stresses at infinity. Hence,