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# A Gurson-type criterion for plastically anisotropic solids containing arbitrary ellipsoidal voids

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ABSTRACT

The aim of this paper is to provide a homogenized criterion for porous ductile materials incorporating both void shape and plastic anisotropy effects. This is done by extending recent criteria of Madou and Leblond (2012a,b) for general ellipsoidal cavities in plastically isotropic matrices, and Monchiet et al. (2008), Keralavarma and Benzerga (2010) for spheroidal cavities in plastically anisotropic matrices, to general ellipsoidal cavities in plastically anisotropic matrices. A limit-analysis is performed of an ellipsoidal representative volume made of some rigid-ideal-plastic Hill material, containing a confocal ellipsoidal void and loaded under conditions of homogeneous boundary strain rate. Use is made in this analysis of some trial incompressible velocity fields discovered by Leblond and Gologanu (2008), satisfying such conditions on an arbitrary family of confocal ellipsoids. Approximations resulting from asymptotic studies of the microscopic plastic dissipation near the void and at infinity lead to an analytic yield function, the coefficients of which are not fully determined at this stage. Complete determination of these coefficients is done using finite element simulations for hydrostatic loadings, on the one hand, and a rigorous bound of Ponte-Castaneda (1991), Willis (1991) and Michel and Suquet (1992) for nonlinear composites for deviatoric loadings, on the other hand.



## 1. Introduction

The most famous model for porous ductile materials is that of Gurson (1977), which was obtained through limit-analysis of a spherical representative cell made of a rigid-ideal-plastic von Mises material, containing a concentric spherical void and subjected to conditions of homogeneous boundary strain rate (Mandel, 1964; Hill, 1967). Due to its intrinsic limitations to spherical voids and plastically isotropic materials, several extensions of this model have been proposed.

Since voids are often non-spherical in real materials, void shape effects have been introduced by Gologanu et al. (1993, 1994, 1997) by considering spheroidal voids. These authors thus performed a limit-analysis of a spheroidal cell made of a von Mises material, containing a confocal spheroidal void and subjected to conditions of homogeneous boundary strain rate.[1] A generalization of these studies has been recently proposed by Madou and Leblond (2012a,b) by considering arbitrary ellipsoidal cavities.

Another type of extensions considered plastically anisotropic matrices obeying Hill (1948)'s orthotropic criterion instead of von Mises's isotropic one. Benzerga and Besson (2001) first performed a limit-analysis of a spherical cell made of a Hill material and containing a spherical void, using Gurson (1977)'s velocity fields. Then Monchiet et al. (2006, 2008) and Keralavarma and Benzerga (2008, 2010) considered spheroidal voids embedded in a Hill matrix, using the velocity fields respectively considered by Gologanu et al. (1993, 1994, 1997), which were discovered by Lee and Mear (1992). All these studies devoted to plastically anisotropic matrices therefore used the same trial velocity fields as those previously considered for the isotropic case.

The aim of this paper is to define a Gurson-type criterion for plastically anisotropic solids obeying Hill (1948)'s criterion, and containing arbitrary ellipsoidal voids. This model will stand as an extension of both Madou and Leblond (2012a,b)'s criterion because the material will be considered to be anisotropic instead of isotropic, and Monchiet et al. (2008), Keralavarma and Benzerga (2010)'s criteria because the voids will be considered to be arbitrary ellipsoidal instead of spheroidal.

To this end, we shall perform a limit-analysis of some general ellipsoidal representative cell made of a rigid-ideal-plastic Hill material, containing a confocal ellipsoidal void and loaded through

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[1] Recently, Monchiet et al. (2014) studied the same problem by means of Eshelby-like velocity fields.

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conditions of homogeneous boundary strain rate. This analysis will use the same trial velocity fields as in the previous work of Madou and Leblond (2012a,b), that is, those discovered by Leblond and Gologanu (2008) and satisfying conditions of this type on an arbitrary family of confocal ellipsoids.

The paper is organized as follows:

- Section 2 presents the limit-analysis based on Madou and Leblond (2012a,b)'s velocity fields. The output is a complex integral expression of the macroscopic plastic dissipation.
- In Section 3, approximations are then made on the macroscopic plastic dissipation, based on asymptotic studies of the microscopic dissipation near the void and at infinity. This leads to an approximate analytical yield function, the coefficients of which are not fully determined at this stage.
- Section 4 presents the explicit determination of the parameters appearing in one term of the yield function, based on finite element micromechanical simulations.
- The parameters of the remainder of the yield function are next determined in Section 5, using a bound for nonlinear composites derived by Ponte-Castaneda, 1991; Willis, 1991; Michel and Suquet, 1992.
- Section 6 provides a summary of all relevant equations for ease of reference.
- Finally Section 7 presents a brief assessment of the yield function through numerical finite element limit analysis.

## 2. Limit analysis of an ellipsoidal cell containing a confocal ellipsoidal void

### 2.1. Limit-analysis procedure

Limit-analysis combined with the Hill–Mandel (Mandel, 1964; Hill, 1967) homogenization theory is a convenient framework to derive constitutive equations for porous ductile solids. Indeed, it permits to effectively operate the scale transition by providing microstructural information in the macroscopic constitutive behavior.

Consider a representative volume element (RVE) in a porous ductile solid denoted $\Omega$ and containing a void denoted $\omega$. The macroscopic yield locus of the porous material can be determined using the upper-bound theorem of limit-analysis (see e.g. Benzerga and Leblond, 2010). The fundamental inequality of this approach

$$\boldsymbol{\Sigma} : \mathbf{D} \leqslant \Pi(\mathbf{D}) \tag{1}$$

leads to the parametric equation of the yield locus

$$\boldsymbol{\Sigma} = \frac{\partial \Pi}{\partial \mathbf{D}}(\mathbf{D}) \tag{2}$$

where the macroscopic stress and strain rate tensors $\boldsymbol{\Sigma}$ and $\mathbf{D}$ are defined as the volume averages of their microscopic counterparts $\boldsymbol{\sigma}$ and $\mathbf{d}$. The macroscopic plastic dissipation $\Pi(\mathbf{D})$ in Eqs. (1) and (2) is defined by:

$$\Pi(\mathbf{D}) = \inf_{\mathbf{v} \in \mathcal{K}(\mathbf{D})} \langle \sup_{\boldsymbol{\sigma}^* \in \mathcal{C}} \boldsymbol{\sigma}^* : \mathbf{d} \rangle_{\Omega-\omega} \tag{3}$$

where $\mathcal{C}$ is the microscopic convex domain of reversibility and the set $\mathcal{K}(\mathbf{D})$ consists of velocity fields $\mathbf{v}$ kinematically admissible with $\mathbf{D}$ and verifying the property of incompressibility.

### 2.2. Presentation of the cell

We consider an ellipsoidal cell containing a confocal ellipsoidal void and loaded arbitrarily through conditions of homogeneous boundary strain rate (Mandel, 1964; Hill, 1967). We briefly recall the notations used by Madou and Leblond (2012a) to describe such a cell.

The semi-axes of the inner ellipsoid (the boundary of the void), parallel to the unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, are denoted $a, b, c$ $(a > b > c)$ while those of the outer one (the boundary of the cell) are denoted $A, B, C$ $(A > B > C)$. These two ellipsoids are related through the confocality conditions $A^2 - a^2 = B^2 - b^2 = C^2 - c^2$. Their volumes are $\frac{4\pi}{3}\omega$ and $\frac{4\pi}{3}\Omega$ where

$$\omega \equiv abc; \quad \Omega \equiv ABC, \tag{4}$$

and the porosity is $f \equiv \frac{\omega}{\Omega}$.

Some elements pertaining to the ellipsoidal coordinated $\lambda, \mu, \nu$ associated to the triplet $(a, b, c)$ are provided in Appendix A. In particular, the coordinate $\lambda$ (which plays the same role as $r$ for spherical coordinates) takes the value $\lambda \equiv 0$ on the inner ellipsoid $\mathcal{E}_0$ and $\lambda \equiv \Lambda$ on the outer one $\mathcal{E}_\Lambda$. The semi-axes $A, B, C$ of $\mathcal{E}_\Lambda$ are related to those, $a, b, c$, of $\mathcal{E}_0$ plus the parameter $\Lambda$ through the relations

$$A \equiv \sqrt{a^2 + \Lambda}; \quad B \equiv \sqrt{b^2 + \Lambda}; \quad C \equiv \sqrt{c^2 + \Lambda}. \tag{5}$$

It then follows from Eqs. (4) and (5) that the parameter $\Lambda$ is determined by the following third-degree polynomial equation:

$$(a^2 + \Lambda)\left(b^2 + \Lambda\right)(c^2 + \Lambda) - \frac{a^2 b^2 c^2}{f^2} = 0. \tag{6}$$

The completely flat ellipsoid confocal with $\mathcal{E}_0$ and $\mathcal{E}_\Lambda$ will play an important role in the sequel. Its semi-axes $\bar{a}, \bar{b}, \bar{c}$ are given by

$$\bar{a} \equiv \sqrt{a^2 - c^2}; \quad \bar{b} \equiv \sqrt{b^2 - c^2}; \quad \bar{c} \equiv 0. \tag{7}$$

The family of confocal ellipsoids $\mathcal{E}_\lambda$ may then be characterized by the single dimensionless parameter

$$k \equiv \frac{\bar{b}}{\bar{a}} = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \tag{8}$$

or the related one

$$k' \equiv \sqrt{1 - k^2} = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}. \tag{9}$$

Note that the ellipsoids $\mathcal{E}_\lambda$ are prolate spheroids if $(k, k') = (0, 1)$, and oblate spheroids if $(k, k') = (1, 0)$.

### 2.3. Hill (1948)'s anisotropic criterion

We assume that the matrix is rigid-plastic and obeys Hill (1948)'s orthotropic yield criterion. The basis of orthotropy is denoted $(\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$ and is *not* supposed to coincide with the principal basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of the cavity.

Let us denote by $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\sigma}}^*$ the Voigt-type vector representations, generally used in finite element codes, of the local stress tensor $\boldsymbol{\sigma}$ in the bases $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and $(\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*)$, respectively:

$$\tilde{\boldsymbol{\sigma}} \equiv \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \\ \tilde{\sigma}_4 \\ \tilde{\sigma}_5 \\ \tilde{\sigma}_6 \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix}; \quad \tilde{\boldsymbol{\sigma}}^* \equiv \begin{pmatrix} \tilde{\sigma}_1^* \\ \tilde{\sigma}_2^* \\ \tilde{\sigma}_3^* \\ \tilde{\sigma}_4^* \\ \tilde{\sigma}_5^* \\ \tilde{\sigma}_6^* \end{pmatrix} \equiv \begin{pmatrix} \sigma_{11}^* \\ \sigma_{22}^* \\ \sigma_{33}^* \\ \sigma_{12}^* \\ \sigma_{13}^* \\ \sigma_{23}^* \end{pmatrix}. \tag{10}$$

Also, let $\tilde{\mathbf{d}}$ and $\tilde{\mathbf{d}}^*$ denote those of the local strain rate tensor $\mathbf{d}$ in the same bases: