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# Three-dimensional acoustic scattering by multiple spheres using collocation multipole method

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ABSTRACT

This paper presents a semi-analytical approach to solve the three-dimensional acoustic scattering problems with multiple spheres subjected to a plane sound wave. To satisfy the three-dimensional Helmholtz equation in a spherical coordinate system, the multipole expansion for the scattered acoustic field is formulated in terms of the associated Legendre functions and the spherical Hankel functions that also satisfy the radiation condition at infinity. The multipole method, the directional derivative and the collocation technique are combined to propose a collocation multipole method in which the acoustic field and its normal derivative with respect to the non-local spherical coordinate system can be calculated without any truncated error, frequently occurred when using the addition theorem. The boundary conditions are satisfied by collocating points on the surface of each sphere. By truncating the higher order terms of the multipole expansion, a finite linear algebraic system is acquired. The scattered field can then be determined according to the given incident sound wave. The convergence analysis considering the specified error, the separation of spheres and the wave number of an incident wave is first carried out to provide guide lines for the proposed method. Then the proposed results for acoustic scattering by one, two and three spheres are validated by using the available analytical method and numerical methods such as boundary element method. Finally, the effects of the separation between scatterers, the incident wave number and the incident angle on the acoustic scattering are investigated extensively.



## 1. Introduction

The subject of acoustic scattering has long attracted the attention of researchers in academia or industry because the results of corresponding studies can be found in many applications (Ingard, 2008) such as locating sound sources, noise control, etc. Although numerical methods such as finite element methods (FEM) and boundary element methods (BEM) (Wu, 2000) can solve these problems, analytical solutions, if available, usually result in accurate and fast-rate convergence methodologies and provide physical insight into the problem under consideration. Furthermore, its results can also provide benchmark solutions that are useful for evaluating the accuracy of various numerical methods. Consequently, a semi-analytical approach to the problem of a plane sound wave scattered by multiple spheres is presented in this work.

During the past few decades, exterior acoustic problems with simple scatterers have been solved by various analytical methods (Bowman et al., 1987; Twersky, 1964; Marnevskaya, 1969, 1970; Gumerov and Duraiswami 2002) or semi-analytical methods (or approximation methods) (Waterman 1969; Peterson and Strom 1974) including the null-field boundary integral equation method (BIEM) (Chen et al., 2010). Rayleigh was the first to obtain and apply the general solutions for scattering of sound by a sphere (Twersky 1964). The Bessel–Legendre series solution was derived to analytically investigate the case of moderately small *ka*. Marnevskaya (1969, 1970) derived the formulation for the problem of diffraction of a plane sound wave by two spheres using the Bessel-Legendre series expansion and the addition theorem for spherical wave functions. Graphs of the far-field scattering intensity versus the spherical angle were presented for the cases where the distance between two spheres is much greater than the wavelength; however, the accuracy of some results is insufficient after careful comparisons. Gumerov and Duraiswami (2002) used the addition theorem of spherical functions to present the multipole reexpansion to solve the problem of multiple scattering from $\boldsymbol{N}$ spheres arbitrary located in three-dimensional space. Waterman (1969) presented a semi-analytical approach, the so called transition (or *T*-) matrix method, for acoustic scattering problem. Peterson and Strom (1974) extended the T-matrix approach to solve the problem with arbitrary number of scatterers. Chen et al. (2010) applied the null-field BIEM to solve radiation and

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scattering problems with multiple spheres. It is well known that the BIEM belongs to the boundary-type method, reducing the dimension of the original problem by the order of one. Consequently the number of the unknowns is much less than that for the domain type methods such as the FEM so that the domain mesh generation, a difficult and time consuming task, can be avoided. However, in addition to the boundary integration required in the BIEM, the BIEM has the problem of singularity and fictitious eigenvalue. Although many remedies were suggested to solve these problems, including the degenerate kernel (Chen et al., 2010) and various algorithms (Chen et al., 2001) to suppress the fictitious eigenvalue, the additional techniques inevitably lead to complicated calculations as well as tedious formulations, limiting their practical applications. Therefore, developing a method with regular characteristics, free of fictitious eigenvalue, is needed.

The multipole method for solving multiply-connected domain problems was firstly proposed by Záviška (1913) and used for the interaction of waves with arrays of circular cylinders by Linton and Evans (1990), the multipole expansion being the so called the wave function expansion. The addition theorem is often employed to transform the multipole expansion into one of the local coordinate systems to satisfy the specified boundary conditions. For the circular boundary, some applications can be seen in the flexural wave scattering (Lee and Chen, 2010). In the case of the sphere, much research can be seen in the literature (Marnevskaya, 1969, 1970; Peterson and Strom, 1974; Gumerov and Duraiswami, 2002). Gumerov and Duraiswami (2005) used the fast multipole method to efficiently solve the large multiple scattering problems. From a mathematical perspective, this procedure is elegant. However, we need to face a difficult formulation due to the infinite series form of the addition theorem, for example spherical wave functions for spheres, limiting its applications.

This paper presents a collocation multipole approach to semi-analytically solve the acoustic scattering problems with multiple spheres. Instead of using the addition theorem, when considering the Neumann boundary conditions (or sound-hard conditions), the normal derivatives of an acoustic pressure with respect to a non-local spherical coordinate system can be exactly calculated by using the directional derivative in each local spherical coordinate system. The given boundary conditions can be satisfied by distributing collocation points on the surface of each sphere. A coupled finite linear algebraic system is derived by truncating the infinite multipole expansion. According to the given incident acoustic wave, the scattering field is obtained through the solution of the algebraic system. Once the total field is calculated as the sum of the incident field and the scattered field, the near-field pressure intensity on the surfaces of scatterers and the far-field scattering pattern can be both determined. The convergence analysis is carried out first to determine the proper number of terms in the multipole expansion for various situations. Then the proposed results are verified by the available analytical method and the numerical methods such as the BEM. Finally the effects of the distance between spheres and the incident wave on the near-field and far-field of behavior of acoustic scattering are investigated extensively. It is worth mentioning that the proposed method is applicable to electromagnetic scattering (Wang and Chew, 1993) or scattering of water wave (Wu, 1995) without any substantial change as spheres are of interest in fields such as electromagnetics and hydrodynamics.

## 2. Problem statement and the general solution in the spherical coordinate system

To properly deal with the geometry considered in this work, the spherical coordinate system shown in Fig. 1 should be used. The spherical coordinates are related the Cartesian coordinates by the relation

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta \tag{1}$$

or

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right) \quad \text{and}$$
$$\phi = \operatorname{atan2}(y/x), \tag{2}$$

where $r \geqslant 0$ is the distance between a field point and the origin, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$.

An unbounded acoustic medium containing $N_s$ spheres subjected to an incident plane sound wave is shown in Fig. 2, where $N_s + 1$ observer coordinate systems are used: $Oxyz$ is a global Cartesian coordinate system and $O_j r_j \theta_j \phi_j$, $j = 1, \ldots, N_s$, is the $j$th local spherical coordinate system, attached to one of the $N_s$ spheres. The position of each of the origins $O_j$ with respect to global Cartesian coordinate system $Oxyz$ is given by $(x^j, y^j, z^j)$. The wave equation for the acoustic pressure $P(r,t)$ in an unbounded homogenous medium is

$$\nabla^2 P(\mathbf{r},t) - \frac{\partial^2 P(\mathbf{r},t)}{\underset{\sim}{c}^2 \partial t^2} = 0, \quad \mathbf{r} \in \Omega^e, \tag{3}$$

where $\nabla^2$ is the Laplace operator, $\underset{\sim}{c}$ is the speed of sound and $\mathbf{r} = (x,y,z)$ is the position of a typical field point in the unbounded exterior region denoted by $\Omega^e$.

For the time-harmonic motion exclusively, solution of Eq. (3) is given by

$$P(\mathbf{r},t) = p(\mathbf{r})e^{-i\omega t}, \tag{4}$$

where $\omega$ is the circular frequency. Hence the complex-valued function $p(\mathbf{r})$ satisfies the following Helmholtz equation,

$$(\nabla^2 + k^2)p(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega^e, \tag{5}$$

where $k = \omega/\underset{\sim}{c}$ is the wave number.

In spherical coordinates, the Helmholtz equation has separated solutions of the form

$$j_n(kr)P_n^m(\mu)e^{im\phi}, \quad y_n(kr)P_n^m(\mu)e^{im\phi}, \quad h_n^{(1)}(kr)P_n^m(\mu)e^{im\phi} \quad \text{and}$$
$$h_n^{(2)}(kr)P_n^m(\mu)e^{im\phi},$$

where the $j_n$ and $y_n$ are called spherical Bessel functions of the first and second kinds, $\mu = \cos\theta$ and $P_n^m$ is the associated Legendre functions of degree $n$ and order $m$. Analogous to the Hankel function, the spherical Hankel function of the first and second kinds are defined by

$$h_n^{(1)}(kr) = j_n(kr) + iy_n(kr) \quad \text{and} \quad h_n^{(2)}(kr) = j_n(kr) - iy_n(kr). \tag{6}$$

Since the functional value of the spherical Bessel function of $y_n$ is infinite at the origin, the permissible solution of Eq. (5) is

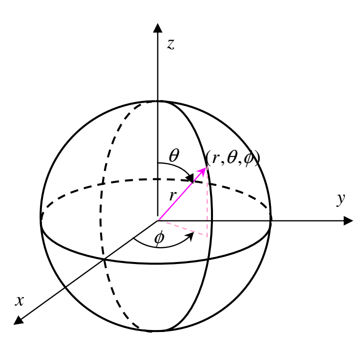


**Fig. 1.** Spherical coordinate system.