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Loss of ellipticity in the combined helical, axial and radial elastic deformations of a fibre-reinforced circular cylindrical tube

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ABSTRACT

In this paper we consider theoretically the finite deformation of a circular cylindrical tube of a transversely isotropic elastic material, specifically the combined axial stretch, inflation and helical shear deformation, with particular reference to the failure of ellipticity. For a simple form of strain-energy function specific examples involving axial and radial directions of transverse isotropy are then considered, leading to different predictions of the onset of ellipticity failure.

1. Introduction

The aim of this work is to analyze the equilibrium configurations of a tube of transversely isotropic hyperelastic material subject to combined axial compression (or extension), inflation and helical shear deformations for which the governing differential equation varies in type locally from strongly elliptic to non-elliptic, or *vice versa*, as the deformation proceeds. This change is associated with the possible emergence of surfaces of weak discontinuity in the deformation, i.e. surfaces on which certain second derivatives of the deformation are discontinuous, sometimes referred to as *weak solutions*, or strong discontinuity where first derivatives of the deformation are discontinuous. For definiteness in this paper we shall use the terminology 'weak solutions' in referring to the discontinuities.

This analysis has been motivated by instability phenomena in fibre-reinforced composite materials. In particular, the material under consideration is an isotropic *neo-Hookean* base (or matrix) material augmented by an energy function that accounts for the existence of fibre reinforcement, and in this work we will deal in particular with the so-called standard model of reinforcement. The loss of ellipticity of the governing differential equations for the considered material is interpreted in terms of fibre failure.

The helical shear problem has been studied by many authors from several points of view in the case of an *isotropic* material, starting from the pioneering work of Rivlin (1949). These include the study of combined axial and azimuthal shear of a circular cylindrical tube of incompressible isotropic elastic material by Ogden et al. (1973), in which some universal relations between the stress components were provided, and the works of Beatty and Jiang (1999) and Kirkinis and Ogden (2003), which were concerned with compressible materials capable of supporting helical shear. Horgan and Saccomandi (2003) investigated different constitutive models that account for hardening at large deformations in the case of a circular cylindrical tube composed of an incompressible hyperelastic material. None of these papers were concerned with the loss of ellipticity, but, by contrast, Fosdick and MacSithigh (1983) provided a detailed study of helical shear with emphasis on the structure of the energy function and its convexity, with particular reference to a non-convex energy function and the emergence of equilibrium configurations with discontinuous deformation gradients.

In the case of an anisotropic material the problem of helical shear has barely been studied, although Jiang and Beatty (2001) derived a necessary and sufficient condition for the strain-energy function to admit helical shear deformations for a compressible, anisotropic hyperelastic circular tube, considering transverse isotropy as a special case. Again, loss of ellipticity was not considered. In terms of loss of ellipticity, Abeyaratne (1981) investigated the emergence of solutions involving discontinuous deformation gradients associated with loss of ellipticity in the finite twisting of an incompressible isotropic elastic tube, while more recently Kassianidis et al. (2008) and Gao and Ogden (2008), from different perspectives, have analyzed the problem of azimuthal shear of a circular cylindrical tube of incompressible *transversely* isotropic elastic material where loss of strong ellipticity and the emergence

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of discontinuous (or non-smooth) solutions was examined. Dorfmann et al. (2010) similarly studied the azimuthal shear problem, but with the anisotropy associated with two symmetrically disposed preferred material directions. The problem of a tube of transversely isotropic elastic material subject to radial and axial deformations coupled with torsion was examined recently by El Hamdaoui et al. (2014) but without reference to loss of ellipticity.

The discussion above relates to deformations of a thick-walled tube. For azimuthal shear, in particular, which is a plane strain deformation, the emergence, development and disappearance of weak surfaces is a feature of the analysis in Kassianidis et al. (2008), Gao and Ogden (2008) and Dorfmann et al. (2010). This is also the case for the three-dimensional helical shear problem which is analyzed herein.

Similar phenomena also arise for the problem of rectilinear shear, which has been studied by Merodio et al. (2007), Destrade et al. (2009) and Baek and Pence (2010). In particular, Merodio et al. (2007) examined the existence of discontinuous solutions associated with fibre kinking for different fibre orientations for the (non-homogeneous) rectilinear shear of a finite thickness slab of transversely isotropic elastic material between two rigid plates. For the standard reinforcing model they obtained a closed-form expression for the amount of shear as a function of the through-thickness coordinate and they showed that fibres were subject to contraction on both sides of a singularity (a kink surface).

Destrade et al. (2009) studied the same problem but with two distinct families of fibres with the shear direction bisecting the directions of the two fibre families. They showed that if the two fibre families have the same mechanical properties then no singularities can arise, but that singularities can develop when one fibre family is significantly stiffer than the other. The paper by Baek and Pence (2010) is concerned with a transversely isotropic material based on the standard reinforcing model, first under simple shear (homogeneous) and then subject to rectilinear shear (inhomogeneous). For simple shear they analyzed in detail the effect of fibre orientation on the emergence, development and disappearance of singular surfaces (kink surfaces) as the shear stress is applied. They went on to extend their analysis to the rectilinear shear problem and found, in particular, that pairs of singular surfaces were nucleated at a critical value of the shear stress and then annihilated at a second critical value, this being associated with a non-monotonic shear stress amount of shear response.

In Section 2 we provide a summary of the basic ingredients of the kinematics and nonlinear elasticity theory, with particular reference to transversely isotropic materials and the loss of ellipticity condition. This is then applied, in Section 3, to a reduced form of the transversely isotropic constitutive law involving one isotropic and one transversely isotropic invariant with two examples of fibre distributions – axial and radial – and it is shown how the emergence or disappearance of singular surfaces depends on the geometrical parameters, the deformation and the strength of the anisotropy. Finally, in Section 4 some concluding remarks are made.

2. Problem formulation

2.1. Kinematics and constitutive laws

We consider a circular cylindrical tube with an undeformed and stress-free reference configuration defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L, \tag{1}$$

where (R, Θ, Z) are cylindrical polar coordinates with associated unit basis vectors $(\mathbf{E}_R,\ \mathbf{E}_\Theta,\ \mathbf{E}_Z)$. The position vector, denoted $\mathbf{X}$, of a material point in this configuration is given by $\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z$ relative to an origin on the tube axis. The deformation of the cylinder is described by the equations

$$r = r(R), \quad \theta = \Theta + g(R), \quad z = \lambda_z Z + w(R), \tag{2}$$

where (r, θ, z) are cylindrical coordinates with unit basis vectors $(\mathbf{e}_r,\ \mathbf{e}_\theta,\ \mathbf{e}_z)$ and $\mathbf{X}$ becomes $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$ (with the same origin). The constant λ_z is the axial stretch of the cylinder, and $g(R)$ and $w(R)$ are unknown azimuthal and axial displacement functions to be determined from the solution of the equilibrium equations and boundary conditions.

The deformation gradient tensor is denoted $\mathbf{F}$ and given by Grad $\mathbf{x}$, where Grad is the gradient operator with respect to $\mathbf{X}$. We assume that the material is incompressible, so that the constraint

$$\det \mathbf{F} = 1 \tag{3}$$

is satisfied. For the considered deformation $\mathbf{F}$ is given by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial R} \otimes \mathbf{E}_R + \frac{1}{R}\frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{E}_\Theta + \frac{\partial \mathbf{x}}{\partial Z} \otimes \mathbf{E}_Z \tag{4}$$

and is calculated explicitly as

$$\mathbf{F} = (\lambda_r \mathbf{e}_r + \gamma_\theta \mathbf{e}_\theta + \gamma_z \mathbf{e}_z) \otimes \mathbf{E}_R + \lambda_\theta \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z \mathbf{e}_z \otimes \mathbf{E}_Z, \tag{5}$$

where $\lambda_r = r'(R)$ is the radial stretch, $\lambda_\theta = r/R$ is the azimuthal stretch, $\gamma_\theta = rg'(R)$ and $\gamma_z = w'(R)$, the prime indicating differentiation with respect to R. Then, by incompressibility,

$$\lambda_r = (\lambda_z \lambda_\theta)^{-1} \tag{6}$$

and

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \tag{7}$$

where $a = r(A)$ is the deformed inner radius of the tube, and we adopt the notation $b = r(B)$ for the outer deformed radius.

The right Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is given by

$$\begin{aligned}\mathbf{C} = {} & (\gamma_z^2 + \gamma_\theta^2 + \lambda_r^2)\mathbf{E}_R \otimes \mathbf{E}_R + \lambda_\theta^2 \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \lambda_z^2 \mathbf{E}_Z \otimes \mathbf{E}_Z \\ & + \gamma_\theta \lambda_\theta (\mathbf{E}_R \otimes \mathbf{E}_\Theta + \mathbf{E}_\Theta \otimes \mathbf{E}_R) + \gamma_z \lambda_z (\mathbf{E}_R \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_R)\end{aligned} \tag{8}$$

and the left Cauchy–Green deformation tensor $\mathbf{F}\mathbf{F}^{\mathrm{T}}$ by

$$\begin{aligned}\mathbf{B} = {} & \lambda_r^2 \mathbf{e}_r \otimes \mathbf{e}_r + (\gamma_\theta^2 + \lambda_\theta^2)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + (\lambda_z^2 + \gamma_z^2)\mathbf{e}_z \otimes \mathbf{e}_z + \gamma_\theta \lambda_r (\mathbf{e}_r \\ & \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + \gamma_z \lambda_r (\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \gamma_\theta \gamma_z (\mathbf{e}_\theta \otimes \mathbf{e}_z \\ & + \mathbf{e}_z \otimes \mathbf{e}_\theta).\end{aligned} \tag{9}$$

We consider an incompressible elastic material with strain-energy function $W(\mathbf{F})$ per unit volume, and by objectivity it depends on $\mathbf{F}$ only through $\mathbf{C}$. The nominal and Cauchy stress tensors $\mathbf{S}$ and $\boldsymbol{\sigma}$ are given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad \boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \tag{10}$$

where p is a Lagrange multiplier associated with the incompressibility constraint and $\mathbf{I}$ is the identity tensor.

In this paper we are concerned with a transversely isotropic material with the direction of transverse isotropy denoted by the unit vector $\mathbf{A}$ in the reference configuration. This can be thought of an isotropic matrix material reinforced by a single family of fibres (with $\mathbf{A}$ the local fibre direction), although this is not essential. For such a material W can be expressed in terms of four invariants in the incompressible case, and in standard notation these are typically taken to be

$$I_1 = \mathrm{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}(\mathbf{C}^2)], \quad I_4 = \mathbf{A}\cdot(\mathbf{C}\mathbf{A}), \quad I_5 = \mathbf{A}\cdot(\mathbf{C}^2\mathbf{A}), \tag{11}$$

where (by incompressibility) $I_3 \equiv \det \mathbf{C} = 1$ has been omitted. In general, $\mathbf{A}$ depends on position $\mathbf{X}$. With $W = W(I_1, I_2, I_4, I_5)$ the Cauchy stress $(10)_2$ expands out in the standard general form