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# Snap-through of arches and buckled beams under unilateral displacement control

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ABSTRACT

Arches and beams buckled upward are analyzed. The structure is pushed downward from above at a specific location along the span until snap-through occurs and the structure jumps to an inverted equilibrium shape. Each beam or arch is modeled as an inextensible elastica. Critical displacements are computed for buckled beams with both ends pinned, both ends clamped, or one end clamped and the other end pinned. Circular arches with pinned ends are also investigated. The ends are immovable. The critical displacement is obtained directly from a theoretical equilibrium shape of the initial unloaded structure. Numerical results are presented for four height-to-span ratios of the initial structure, showing the critical displacement for any application point along the span. At the onset of snap-through, the imposed displacement is at or below the horizontal chord connecting the ends.



## 1. Introduction

Snap-through of arches and buckled beams that are pushed downward at a point along the span is investigated. The structure is free to move downward at that point, but not upward, which is unilateral (one-way) displacement control. After being pushed a certain distance, the structures considered here snap downward to an inverted configuration. For various height-to-span ratios of the initial structure, the critical displacement is plotted as a function of the push-down location.

Snap-through caused by force control has received much more attention in the literature than displacement control. Under force control related to the problem treated here, a concentrated downward force would be increased until snap-through occurs, which is associated with a critical point (bifurcation or limit point) on the equilibrium path (force versus displacement). The dependence of the critical force on its location has been examined in several papers, e.g., Plaut (1979) for shallow extensible arches with the three standard sets of end conditions, Camescasse et al. (2013, 2014) for shallow extensible buckled beams with pinned ends, Fargette et al. (2014) for a shallow inextensible buckled beam with clamped ends, and Harvey and Virgin (2015) for shallow inextensible buckled beams with pinned ends.

Some recent papers have examined snap-through under displacement control, but have not presented results revealing the dependency of the critical displacement on location. They include Cazottes et al. (2009), Chen and Hung (2011), Fargette et al. (2014), Pandey et al. (2014) and Harvey and Virgin (2015).

It is noted that for the unilateral displacement-control problem treated here, the critical displacement is computed from a wavy equilibrium shape of the initial *unloaded* structure. Therefore only the unloaded arch or buckled beam needs to be analyzed. It is not necessary to calculate an equilibrium path. Also, it is noted that snap-through in the present problem occurs after snap-through would have been exhibited under force control, and that part of the structure lies below the horizontal when it snaps to a completely inverted configuration in the cases treated here.

The problem is formulated in Section 2. Results are presented in Sections 3, 4, and 5, respectively, for buckled beams with pinned ends, clamped ends, and one end clamped and the other pinned. Circular arches with pinned ends are analyzed in Section 6, followed by concluding remarks in Section 7.

## 2. Formulation

Linearly elastic, uniform arches and buckled beams are considered, with each end immovable and either pinned or clamped (fixed). The total arc length is $L$, the span (base length) is $B$, the initial height is $H$, and the constant bending stiffness is $EI$. The origin of the coordinate system is at the left end, the $X$ axis is horizontal along the chord connecting the two ends, the $Y$ axis is vertical (positive if upward), and the arc length is $S$. The structure is pushed downward at the location $X = C$, where the vertical distance from

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the horizontal is $D$, positive if upward, and the arc length is $A$. The downward force associated with the displacement control at $X = C$ is $F$.

The following nondimensional quantities are utilized in the rest of this paper:

$$s = S/B,\quad x = X/B,\quad y = Y/B,\quad a = A/B,\quad c = C/B,\quad d = D/B,$$
$$h = H/B,\quad \ell = L/B,\quad m = MB/(EI),\quad f = FB^2/(EI),\quad p = PB^2/(EI),$$
$$q = QB^2/(EI) \tag{1}$$

Fig. 1 depicts a sketch of the structure in nondimensional terms if the ends are pinned. The span is unity.

At location $s$, the coordinates are $x(s)$ and $y(s)$, and the angle between the horizontal and the tangent is $\theta(s)$. At the left end, $\theta(0)$ is denoted $\alpha$. On the positive face of the cross section at $s$, the internal horizontal force is $p(s)$, positive if compressive, the internal vertical force is $q(s)$, positive if downward, and the bending moment is $m(s)$, positive if counter-clockwise.

As the structure is pushed downward at $x = c$, the vertical displacement $d$ there decreases, while the associated force $f$ at $x = c$ increases from zero and then decreases. When $f$ reaches zero, snap-through occurs (Fargette et al., 2014; Harvey and Virgin, 2015) and the structure jumps to an inverted equilibrium configuration, assuming that the system is damped. The vertical position $d$ at the onset of snap-through is denoted the critical displacement $d_{cr}$. Plots of $d_{cr}$ versus $c$ will be presented for total arc lengths $\ell = 1.05$, 1.1, 1.15, and 1.2.

The structure is modeled as an inextensible elastica. Its weight is neglected, and a quasi-static analysis is conducted. For the buckled beams, which are unstrained when straight, the governing equations for $0 < s < \ell$, based on geometry, constitutive law, and equilibrium, are (Plaut and Virgin, 2014)

$$x'(s) = \cos\theta(s),\quad y'(s) = \sin\theta(s),\quad \theta'(s) = m(s),$$
$$m'(s) = q(s)\cos\theta(s) - p(s)\sin\theta(s) \tag{2}$$

The modification of Eq. (2c) for arches will be discussed in Section 6. If the left end of the structure is pinned, $x(0) = y(0) = m(0) = 0$. If it is clamped, $x(0) = y(0) = \theta(0) = 0$. Similar conditions hold at the right end where $s = \ell$.

Numerical solutions are obtained with the use of a shooting method, utilizing the subroutines NDSolve and FindRoot in Mathematica (Plaut and Virgin, 2014). Different equilibrium solutions can be found with the use of different initial guesses for the unknown quantities in the shooting procedure. When $f = 0$, as in the initial and inverted equilibrium shapes and in the wavy equilibrium shape at the onset of snap-through, $p(s)$ and $q(s)$ are constant, with $q = 0$ if the ends are pinned (Sections 3 and 6) or if the ends are clamped and the equilibrium shape is symmetric (initial and inverted shapes in Section 4).

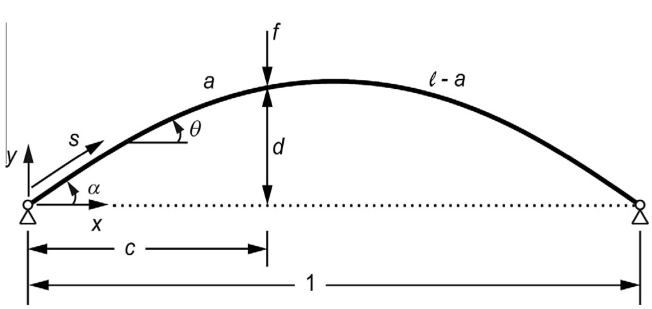


**Fig. 1.** Schematic in nondimensional terms of arch or buckled beam with pinned ends.

## 3. Pinned–pinned buckled beam

A beam with pinned ends is considered in this section. One end is moved toward the other, and the beam is assumed to buckle upward. Then the ends are constrained to be immovable. The ratio of the total arc length to the span is $\ell$ (corresponding to end-shortening $\ell - 1$). This is the initial buckled beam to which the displacement control is applied.

For the case $\ell = 1.1$ and no applied force or imposed displacement, the upper shape in Fig. 2 depicts the initial beam, the dashed shape shows the anti-symmetric equilibrium shape that is below the horizontal (the dotted line) on the left half, and the lower shape is the inverted equilibrium shape. (It is noted that these shapes are not the classical buckling modes of a pinned–pinned column, which are only applicable at the onset of buckling from a straight state.)

In comparing the dashed shape in Fig. 2 to the initial (upper) shape, its axial compressive load $p$ is four times as high, the magnitude of its slope $\alpha$ at the left end is the same, the magnitude of its bending moment at the quarter point $x = 1/4$ is twice that of the initial shape at the midpoint $x = 1/2$, and the magnitude of its height at the quarter point is half the central height $h$ of the initial shape.

If the buckled beam is pushed down at a location $x = c$ on the left half ($0 < c \leqslant 0.5$), the critical displacement $d_{cr}$ is given by the value of the dashed line in Fig. 2 at $x = c$ if $\ell = 1.1$, and by similar curves for other values of $\ell$. Due to symmetry, the reflection of the left curve across the center (i.e., the mirror image) furnishes the curve of $d_{cr}$ versus $c$ for $0.5 \leqslant c < 1$.

Plots of $d_{cr}$ versus $c$ are presented in Fig. 3 for $\ell = 1.05$, 1.1, 1.15, and 1.2. The corresponding initial heights of the buckled beams are $h = 0.144$, 0.205, 0.253, and 0.295. If the beam is pushed down at its midpoint ($c = 0.5$), snap-through occurs when $d_{cr} = 0$, i.e., when the beam's midpoint reaches the horizontal, as previously seen in Harvey and Virgin (2015) and for clamped–clamped buckled beams in Pandey et al. (2014). The minimum values of $d_{cr}$ in Fig. 3 for $\ell = 1.05$, 1.1, 1.15, and 1.2, respectively, are $-0.072$, $-0.103$, $-0.127$, and $-0.148$, which (as for all values of $\ell$) are equal to $-h/2$ and occur at $c = 1/4$ and 3/4.

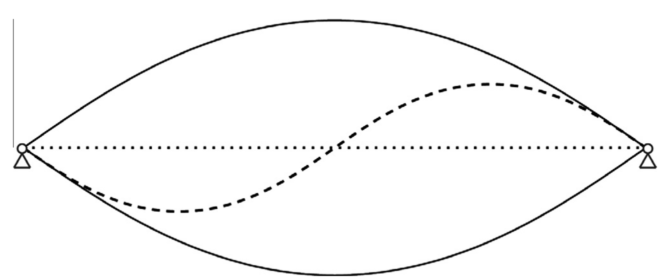

**Fig. 2.** Three equilibrium shapes of pinned–pinned buckled beam for $\ell = 1.1$ and $f = 0$.

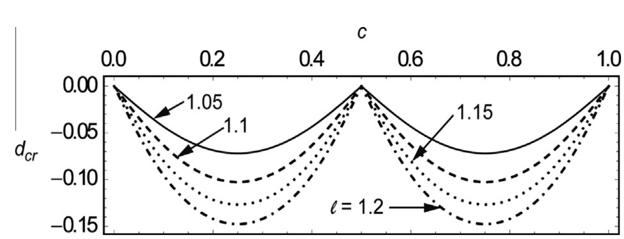


**Fig. 3.** Critical displacement $d_{cr}$ versus push-down location $c$ for pinned–pinned buckled beam with $\ell = 1.05$ (solid), 1.1 (dashed), 1.15 (dotted), and 1.2 (dot-dashed).