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The dimensional reduction approach for 2D non-prismatic beam modelling: A solution based on Hellinger–Reissner principle

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ABSTRACT

The present paper considers a non-prismatic beam i.e., a beam with a cross-section varying along the beam axis. In particular, we derive and discuss a model of a 2D linear-elastic non-prismatic beam and the corresponding finite element. To derive the beam model, we use the so-called dimensional reduction approach: from a suitable weak formulation of the 2D linear elastic problem, we introduce a variable cross-section approximation and perform a cross-section integration. The satisfaction of the boundary equilibrium on lateral surfaces is crucial in determining the model accuracy since it leads to consider correct stress-distribution and coupling terms (i.e., equation terms that allow to model the interaction between axial-stretch and bending). Therefore, we assume as a starting point the Hellinger–Reissner functional in a formulation that privileges the satisfaction of equilibrium equations and we use a cross-section approximation that exactly enforces the boundary equilibrium.

The obtained beam-model is governed by linear Ordinary Differential Equations (ODEs) with non-constant coefficients for which an analytical solution cannot be found, in general. As a consequence, starting from the beam model, we develop the corresponding beam finite element approximation. Numerical results show that the proposed beam model and the corresponding finite element are capable to correctly predict displacement and stress distributions in non-trivial cases like tapered and arch-shaped beams.

1. Introduction

Non-prismatic beams are slender bodies in which the position of the cross-section barycentre, the cross-section shape, and/or the cross-section size vary along the prevalent dimension of the body. Those bodies are widely used in engineering practice since they provide effective solutions for optimization problems. As an example, arc-shaped beams (in Fig. 1(a) the Risorgimento bridge, Verona, Italy) could be optimized in order to carry the loads using the minimum amount of materials. As an other, more sophisticated example, windmill turbine blades (in Fig. 1(b) the fiberglass-reinforced epoxy blades of Siemens SWT-2.3–101 wind turbines) are optimized with respect to different conflicting needs like aerodynamic efficiency, noise pollution, forces induced on the tower. The models that describe the behavior of non-prismatic beams must be as efficient as possible in order to perform an effective design. Unfortunately, non-prismatic beam models rarely satisfy the needs of the practitioners, who must choose between refined but too expensive models –like 3D Finite Element (FE) analysis– and inexpensive but too coarse models –like frame analysis that uses 1D elements with piecewise-constant cross-sections.

Consider first the tapered beams, i.e. a class of non-prismatic beams with the following properties: (i) the beam has a straight axis, (ii) the cross-section dimension varies linearly with respect to the axis coordinate, and (iii) the cross sections have at least two symmetry axes whose intersection coincides with the beam axis. Under these conditions, the positions of either cross-section barycentre (i.e., the point where a resulting axial force can be applied without inducing any bending moment) and shear-centre (i.e., the point where a resulting shear force can be applied without inducing any torsion) do not depend on the beam-axis coordinate. The tapered-beam modeling takes advantage of the tapered-beam geometry since it ensures that axial-, transverse-, and rotation-equilibrium equations are independent. As a consequence of their simplicity, tapered beams are deeply investigated and many modeling approaches have been proposed in the literature, as illustrated in the following. The simplest modeling-approach consists in modifying the coefficients of the Euler–Bernoulli (EB) or Timoshenko beam-model equations in order to take into account the variation of the cross-section area and inertia along the beam

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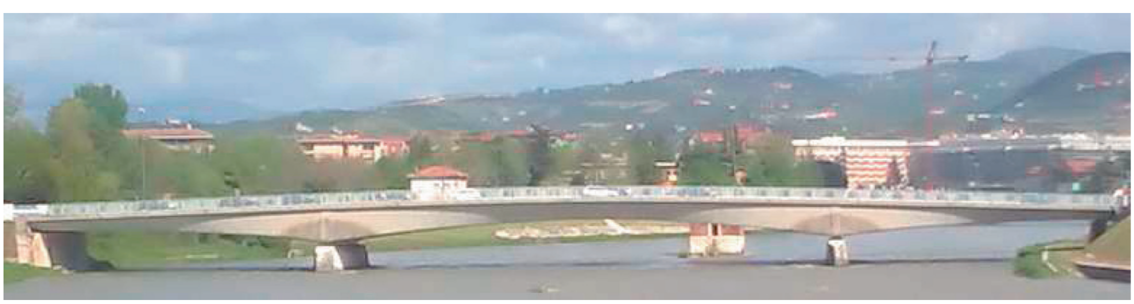
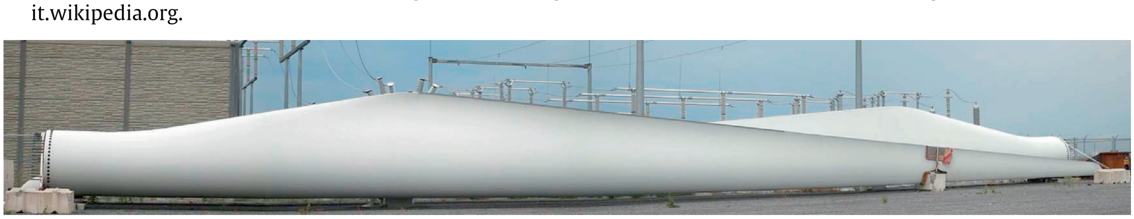
Nomenclature

E	Young's modulus
$H(0), H(\bar{l})$	initial and final cross-sections
$H(x)$	beam cross-section
J_{HR}	Hellinger–Reissner (HR) functional
$L^2(\Omega), H(\mathrm{div}, \Omega)$	2D Sobolev spaces
$L^2(l), H^1(l)$	beam-model Sobolev spaces
$M(x)$	bending moment
$N(x)$	resulting axial stress
O, x, y	Cartesian coordinate system
$V(x)$	resulting shear
W, S_0, S_t	2D HR functional spaces
Δ	difference of cross-section height
Ω	beam body i.e., 2D problem domain
$\delta\boldsymbol{s}, \delta\boldsymbol{\sigma}$	virtual fields
$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$	x- and y- partial derivatives
γ	generic field
γ^{ref}	reference solution
$\hat{\boldsymbol{\gamma}}$	axial coefficient functions
$\hat{\boldsymbol{t}}_x, \hat{\boldsymbol{t}}_y$	projection of external load on profile functions
λ	wave length
$\nabla \cdot (\cdot)$	divergence operator
ν	Poisson's coefficient
$\overline{H}(x)$	cross-section height
$\bar{\boldsymbol{s}}$	boundary displacement function
$\bar{l}$	beam length
$\overline{u}, \overline{v}$	cross-section axial- and transversal- displacement mean-values
$\partial\Omega$	domain boundary
$\partial\Omega_s, \partial\Omega_t$	displacement constrained and externally loaded boundaries
$\boldsymbol{D}$	fourth-order elastic tensor
$\boldsymbol{E}_1, \boldsymbol{E}_2$	engineering notation's Boolean matrices
$\boldsymbol{F}$	beam-model load vector
$\boldsymbol{G}, \boldsymbol{H}$	ODE coefficient matrices
$\boldsymbol{H}_{\sigma s}, \boldsymbol{H}_{\sigma\sigma}, \boldsymbol{G}_{\sigma s}$	beam-model coefficient matrices
$\boldsymbol{K}_{s\sigma}, \boldsymbol{K}_{\sigma\sigma}$	FE stiffness matrices
$\boldsymbol{N}_{\gamma i}$	axis shape functions
$\boldsymbol{P}_s, \boldsymbol{P}_\sigma$	matrices collecting displacement and stress profile functions
$\boldsymbol{R}$	matrix accounting for boundary equilibrium
$\boldsymbol{T}$	beam-model external load vector
$\boldsymbol{\sigma}$	symmetric stress tensor field
$\boldsymbol{f}$	distributed load
$\boldsymbol{n}$	outward unit vector
$\boldsymbol{p}_\gamma$	profile functions
$\boldsymbol{s}$	displacement vector field
$\boldsymbol{t}$	external load distribution
σ_x, σ_y, τ	axial, transversal, and shear stresses
$\widetilde{W}, \widetilde{S}$	beam-model variational formulation spaces
$\widetilde{\boldsymbol{T}}$	FE load vector
$\widetilde{\gamma}_i$	numerical coefficients
$e(x)$	eccentricity
e_γ^{rel}	relative error
$h_l(x), h_u(x)$	cross section lower- and upper- boundaries
l	beam longitudinal axis
m	number of profile functions
u, v	horizontal and vertical displacements
t	number of axis shape functions



(a) Risorgimento bridge, Verona, Italy, designed by eng. Pier Luigi Nervi (1963). The cross-section shape changes in order to maximize the resistance to the bending moment using the minimum amount of material. Image from it.wikipedia.org.



(b) Fiberglass-reinforced epoxy blades of Siemens SWT-2.3-101 wind turbines. Image from it.wikipedia.org.

Fig. 1. Examples of structures that could be seen as non-prismatic beams.

axis. Banerjee and Williams (1985, 1986) illustrate significant examples of this modeling approach, used for example in Vinod et al. (2007) as the basis of the FE analysis. Unfortunately, it is well-known that this approach introduces a modeling error proportional to the rate of cross-section size change which is non-negligible also for small rates (see Boley, 1963). Moreover, investigating the effect of the variation of cross-section size, Hodges et al. (2010) show that the model degeneration is a consequence of the violation of the boundary equilibrium on the lateral surface in the beam model formulation.