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Analytical bounds for damage induced planar anisotropy

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ABSTRACT

The damage of a planar, elastic and initially isotropic material is considered in the framework of a classical approach where the damaged elasticity tensor is ruled by a fourth-rank symmetric damage tensor. The analysis is completely carried on using the so-called polar method for the invariant representation of tensors in $\mathfrak{R}^2$. The final elastic behavior, induced by damage, can be anisotropic: all the possible situations of elastic symmetries are considered, and for each one an analytical expression for the bounds on the invariants of the damaged elastic tensor and of the damage tensor is given. An admissible domain for the damage invariants and for the damaged elastic invariants is so provided, the convexity of these domains is also proved.

1. Introduction

We consider in this paper the anisotropy induced by damage on an initially isotropic layer. The goal is twofold: first, if the elastic tensor of the virgin material is $\mathbb{C}$, determine which is the final tensor $\widetilde{\mathbb{C}}$. Then, to give explicit bounds for the elastic moduli of $\widetilde{\mathbb{C}}$ and for the characteristics of the damage tensor $\mathbb{D}$. To this purpose, we define the damage tensor $\mathbb{D}$ as a fourth-rank tensor with minor and major tensor symmetries, such that the elastic tensor $\widetilde{\mathbb{C}}$ of the damaged material linearly depends upon $\mathbb{C}$ and $\mathbb{D}$, Chaboche, 1978; Chaboche, 1979; Leckie and Onat, 1980; Sidoroff, 1980; Chow, 1987; Lemaitre et al., 2009:

$$\widetilde{\mathbb{C}} = [(1-\mathbb{D})\mathbb{C}]^{Sym} \Rightarrow \widetilde{\mathbb{C}} = \mathbb{C} - \widehat{\mathbb{C}} \quad \text{with } \widehat{\mathbb{C}} = \frac{\mathbb{C}\mathbb{D}+\mathbb{D}\mathbb{C}}{2}. \tag{1}$$

The elastic tensor $\mathbb{C}$ of the virgin material and the damaged elastic tensor $\widetilde{\mathbb{C}}$ must be positive definite, as a consequence of the positiveness of the elastic potential. In a thermodynamical framework, the positive semi-definiteness of the loss of stiffness tensor $\widehat{\mathbb{C}}$ is equivalent to a positive intrinsic dissipation due to linear elasticity-damage coupling (more details are given in Section 4 and Appendix B). The damage tensor $\mathbb{D}$ is assumed to be positive semi-definite.

The conditions of positive semi-definiteness for $\mathbb{D}$ and $\widehat{\mathbb{C}}$ and positive definiteness for $\widetilde{\mathbb{C}}$ provide the conditions to determine the bounds on the values of their moduli, once those on $\mathbb{C}$ known. The problem of whether some of these conditions are more restrictive than the others will be solved in the present work, by showing that the positive semi-definiteness of $\widehat{\mathbb{C}}$ always implies the positive semi-definiteness of $\mathbb{D}$, and is even equivalent in some particular cases related to the induced anisotropy by damage.

To investigate this problem, we make use of the so-called *polar formalism* (Verchery, 1979; Vannucci, 2005). This method gives a representation of elasticity based upon tensor invariants and the different elastic symmetries are readily identified by the values taken by some of these invariants.

We obtain the polar invariants of $\widetilde{\mathbb{C}}$ as functions of those of $\mathbb{C}$ and $\mathbb{D}$; while we assume that the initial material is isotropic, we consider all the possible transformations for the damaged material, leading to a final elastic behavior that can be completely anisotropic, orthotropic, specially orthotropic or also isotropic.

Then, we pass to consider the bounds that damage process impose to the polar moduli of $\mathbb{D}$ and $\widetilde{\mathbb{C}}$; starting from the simpler case, that of an isotropic tensor $\widetilde{\mathbb{C}}$, we consider all the possible cases of elastic symmetries for $\widetilde{\mathbb{C}}$, until the most general case of complete anisotropy. We give an explicit expression for these bounds and show that the admissible domain for the moduli is convex in all the cases, in some of them a graphical representation is also possible.

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2. Essentials of the polar formalism

In this section, we briefly recall the essentials, for the present paper, of the polar formalism. The polar method is basically a mathematical method to search for a complete set of the invariants of a given tensor in $\mathfrak{R}^2$. As such, it can be applied not only to elasticity tensors, but also to any other plane tensor, see for instance Vannucci (2007) or Vannucci and Verchery (2010). The polar formalism is based upon a complex variable method, a technique once widely used in physical mathematics and which has its initiators in the pioneer works of Michell (1902) and Kolosov (1909) and which finds its completion in the treatises of Muskhelishvili (1953), Green and Zerna (1954) and Milne-Thomson (1960). As a consequence, the polar formalism can be applied only to plane problems.

Verchery makes use, just like Green and Zerna, of a complex variable transformation, interpreted as a change of frame; this transformation has some algebraic properties that allows simplifying the expressions of frame rotations and symmetries, which renders rather easy the search for tensor invariants. The mathematical details and passages, rather technical and not to be detailed here, can be found in Vannucci (2005) and Vannucci and Verchery (2010), whereto the interested reader is addressed for a complete explanation of the method.

Here, we recall just the main features of the polar method for a plane fourth-rank tensor $\mathbb{T}$ owing the minor and major tensor symmetries, i.e. such that $\forall i, j, k, l = 1, 2$,

$$\begin{aligned} T_{ijkl} &= T_{jikl} = T_{ijlk}, \\ T_{ijkl} &= T_{klij}. \end{aligned} \tag{2}$$

Five invariants suffice to completely describe $\mathbb{T}$, because in any frame it is completely described by six quantities. Among them, one is needed to fix a frame, so the remaining five ones can be reduced to independent tensor invariants. In the polar formalism, four of these invariants are elastic moduli and are indicated by the symbols T_0, T_1, R_0 and R_1; the last invariant is the angular difference $\Phi_0 - \Phi_1$. One of the two angles, Φ_0 or Φ_1, may be arbitrarily chosen to fixe a reference frame (the most usual choice is $\Phi_1 = 0$).

The basic result of the polar formalism is the expression of the Cartesian components of $\mathbb{T}$ in terms of the polar parameters, in a frame rotated through an angle θ:

$$\begin{aligned} T_{1111}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) + 4R_1 \cos 2(\Phi_1 - \theta), \\ T_{1112}(\theta) &= R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\ T_{1122}(\theta) &= -T_0 + 2T_1 - R_0 \cos 4(\Phi_0 - \theta), \\ T_{1212}(\theta) &= T_0 - R_0 \cos 4(\Phi_0 - \theta), \\ T_{1222}(\theta) &= -R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\ T_{2222}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) - 4R_1 \cos 2(\Phi_1 - \theta). \end{aligned} \tag{3}$$

The above relations show that T_0 and T_1 are the isotropy invariants, while anisotropy is described by the invariants R_0, R_1 and $\Phi_0 - \Phi_1$. In particular, it is easy to recognize that for isotropic elasticity,

$$T_0 = G, \quad T_1 = \frac{1}{2}\kappa, \tag{4}$$

where G and κ are respectively the shear and bulk moduli. The same physical meaning is preserved also for all the anisotropic cases, so we can consider T_0 and T_1 as a generalization, to any elastic behavior, of the shear and bulk moduli, respectively. We remark also that

$$T_0 + 2T_1 = G + \kappa, \tag{5}$$

a quantity often appearing in the following.

The relations giving the polar components as functions of the Cartesian ones can be obtained inverting Eqs. (3):

$$\begin{aligned} 8T_0 &= T_{1111}(\theta) - 2T_{1122}(\theta) + 4T_{1212}(\theta) + T_{2222}(\theta), \\ 8T_1 &= T_{1111}(\theta) + 2T_{1122}(\theta) + T_{2222}(\theta), \\ 8R_0 e^{4i(\Phi_0 - \theta)} &= T_{1111}(\theta) - 2T_{1122}(\theta) - 4T_{1212}(\theta) \\ &\quad + T_{2222}(\theta) + +4i[T_{1112}(\theta) - T_{1222}(\theta)], \\ 8R_1 e^{2i(\Phi_1 - \theta)} &= T_{1111}(\theta) - T_{2222}(\theta) + 2i[T_{1112}(\theta) + T_{1222}(\theta)]. \end{aligned} \tag{6}$$

Denoting by lower-case letters the polar parameters of $\mathbb{T}^{-1}$, it is:

$$\begin{aligned} t_0 &= \frac{2}{\Delta}\left(T_0 T_1 - R_1^2\right), \\ t_1 &= \frac{1}{2\Delta}\left(T_0^2 - R_0^2\right), \\ r_0 e^{4i\varphi_0} &= \frac{2}{\Delta}\left(R_1^2 e^{4i\Phi_1} - T_1 R_0 e^{4i\Phi_0}\right), \\ r_1 e^{2i\varphi_1} &= -\frac{1}{\Delta} R_1 e^{2i\Phi_1}\left[T_0 - R_0 e^{4i(\Phi_0 - \Phi_1)}\right], \end{aligned} \tag{7}$$

with

$$\Delta = 8T_1\left(T_0^2 - R_0^2\right) - 16R_1^2[T_0 - R_0 \cos 4(\Phi_0 - \Phi_1)]. \tag{8}$$

There is a close relation between the polar invariants and elastic symmetries. In particular, the polar analysis of elastic symmetries let appear an *algebraic* characterization of the elastic symmetries, for some aspects more powerful than the classical *geometrical* characterization using the symmetry of the elastic properties linked to a subjacent symmetric distribution of the matter. In fact, special values taken by one ore two invariants determine an elastic symmetry, and these particular values affect and characterize the properties of the matter. Upon this consideration, it can be shown that there are five different and non equivalent types of planar elastic symmetries:

- *ordinary orthotropy*: it corresponds to the condition

$$\Phi_0 - \Phi_1 = K\frac{\pi}{4}, \quad K \in \{0, 1\}; \tag{9}$$

  As a consequence, it is possible to have, for the same set of invariant polar moduli T_0, T_1, R_0 and R_1, two different orthotropic materials, one with $K = 0$, the other one with $K = 1$, whose properties are quite different. These two types of ordinary orthotropic materials correspond to those termed by Pedersen (1989) as *low*, $K = 0$, and *high*, $K = 1$, shear modulus orthotropy. The results obtained by Vannucci (2009), Vincenti and Desmorat (2011), Catapano et al. (2012) and Barsotti and Vannucci (2013) suggest that such a classification is rather restrictive: the differences between these two classes are not limited to shear, but rather concern the overall mechanical response of the material. This can be observed also for the effects of damage, as it will be clear in the following of this paper;
- R_0-*orthotropy*:

$$R_0 = 0; \tag{10}$$

  in this case, the Cartesian components of $\mathbb{T}$ are either constant or change, after a rotation, as those of a second- and not of a fourth-rank tensor, Vannucci (2002); the existence of this special case of orthotropy has been successively found also in $\mathfrak{R}^3$, Forte (2005); a sufficient condition for having R_0-orthotropy is to strengthen (or weaken) an isotropic layer by fibers (cracks) that are shifted of $\pi/4$;
- r_0-*orthotropy*:

$$r_0 = 0; \tag{11}$$

  as Eq. (7_3) clearly shows, condition (10) does not imply the same result for $\mathbb{T}^{-1}$: R_0-orthotropy does not concern the inverse