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Closed-form expressions for the effective moduli of heterogeneous piezoelectric materials

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ABSTRACT

In this paper, analytical and semi-analytical expressions for the effective properties of transversely isotropic piezoelectric materials are derived based on the Mori–Tanaka model and Eshelby tensors using compact matrix formulations. Concise explicit relations are presented for the coupled and decoupled effective electroelastic moduli of piezoelectric materials with various inclusion types and shapes. In order to use these explicit relationships, some of the Eshelby tensor components are needed to be analytically or numerically computed. These closed-form relations are suitable for use in designing composite piezoelectric materials with optimized characteristics. Results of effective electroelastic moduli, obtained using the derived closed-form expressions, are presented for various types and shapes of piezoelectric inclusions and compared with self-consistent and incremental self-consistent numerical predictions.

1. Introduction

Piezoelectric materials are excellent candidates for use in sensors and actuators because of their ability to couple electrical and mechanical energy. The characterization of piezoelectric composites is very important for their production and application. To facilitate the design of such piezoelectric composite systems, convenient and accurate structure–property relationships are needed. The primary motivation of this paper is to develop simple and concise analytical relations for some effective properties of reinforced piezoelectric materials with inclusions of various types and shapes.

Over the past few decades, widespread attention has been paid to composite piezoelectric materials. Considerable effort has been directed toward the analysis and prediction of the effective properties of piezoelectric composites that contain reinforcements. Numerous attempts have been made to develop models that relate the effective electroelastic properties of composite materials to those of the materials' individual constituents. Eshelby (1957) introduced the concept of "equivalent inclusion" for solving the transformation problem when the matrix and the region of the inhomogeneity have different elastic constants and solved for the elastic stress field in and around an ellipsoidal particle using an infinite matrix. For piezoelectric composites, Newnham et al. (1978) proposed a theory based on the idealized parallel and series connections of the constituents based on their three-dimensional connectivity characteristics to analyze the effective electroelastic moduli of piezoelectric composites. Various models have been used to predict the behavior of a limited class of composites. A theoretical model has been presented for ultrasonic composite materials (Chan and Unsworth, 1989). The effective piezoelectric coefficients of isotropic polycrystals with piezoelectric grains have been studied using an effective-medium approximation (Olson and Avellaneda, 1992). The variational formulation has been used by Gaudenzi (1997) to compute the coupled electromechanical response of an active composite material. Composites with piezoceramic fibers have been studied by Poizat and Sester (1999). Benveniste (1992) obtained the coupled electroelastic field for the problem of a single ellipsoidal inclusion without providing any estimation of the average fields. Several authors have extended Eshelby's classical solution for elasticity to piezoelectric material. Piezoelectric Eshelby tensors have been established by Deeg (1980), Dunn (1994), Huang and Yu (1994) and Mikata (2000, 2001). Predictions of the elastic, dielectric and piezoelectric moduli of two-phase composite materials reinforced by ellipsoidal particles or fibers have been presented (Dunn and Taya, 1993a,b;

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Chen, 1994). Huang and Kuo (1996) have used the Mori–Tanaka' method to determine analytical expressions for the effective electroelastic properties of composites in terms of phase properties, orientation angles, volume fractions and inhomogeneity shapes. Various micromechanical models such as Mori–Tanaka, dilute, self-consistent and incremental self-consistent models have been elaborated to describe the effective properties of multiphase electroelastic materials (Fakri et al., 2003, 2010) and magnetoelectro-elastic heterogeneous materials (Bakkali et al., 2011, 2013). Some of these previously cited micromechanical models are based on numerical procedures for which computational codes are needed to be elaborated. In this paper, concise explicit analytical relationships are derived for some of the effective piezoelectric material constants in terms of Eshelby tensor components based on Mori–Tanaka approach. In other words, some of the matrix computations are explicitly analytically evaluated, instead of numerically.

In this paper, mathematical formulas are derived for the coupled and decoupled effective coefficients of transversely isotropic piezoelectric materials based on the Mori–Tanaka model, Eshelby tensors and matrix formulations. These closed forms provide explicit and simple expressions for some piezoelectric moduli. In order to utilize these explicit relationships, some of the Eshelby tensor components are needed to be analytically or numerically computed. For some cases, closed-form expressions are analytically presented. These models can be used to easily predict the effective properties of reinforced piezoelectric materials with various types and shapes of inclusions, provided that the required piezoelectric Eshelby tensor components are given. Predicted results for various effective electroelastic coefficients are presented and compared with the numerical results obtained using the self consistent and incremental self consistent models.

2. Mathematical formulation

2.1. Generalized tensor notations

For linear piezoelectric materials, the coupled interactions between the electrical and mechanical variables are expressed by the following basic equations:

$$\sigma_{ij} = C_{ijmn}.\varepsilon_{mn} - e_{nij}.E_n \quad (1)$$

$$D_i = e_{imn}.\varepsilon_{mn} + \kappa_{in}.E_n \quad (2)$$

where the elastic strain ε_{mn} and electric field E_n are independent variables and are related to the stress σ_{ij} and the electric displacement $D_i \cdot C_{ijmn}, e_{imn}$ and κ_{in} are the elastic moduli (measured in a constant electric field), the piezoelectric coefficients (measured at a constant strain or electric field) and the dielectric constants (measured at a constant strain) respectively. The strain and electric field can be derived from the elastic displacement vector u and the electric potential ϕ using the compatibility equations as follows:

$$\varepsilon_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m}) \quad (3)$$

$$E_n = -\phi_{,n} \quad (4)$$

To complete the formulation of the stationary theory of piezoelectricity, Eqs. (1)–(4) must be supplemented with the equations of elastic equilibrium and Gauss's law of electrostatics which in the absence of any body forces or free charge, are

$$\sigma_{ij,j} = 0 \quad (5)$$

$$D_{i,i} = 0 \quad (6)$$

In the modeling of piezoelectric materials, it is more convenient to restate Eq. (1) such that the elastic and electric variables are combined to yield a single constitutive equation. This notation is identical to the conventional indicial notation with the exception that lower-case subscripts retain the range of 1–3 whereas capitalized subscripts take on the range of 1–4, with repeated capitalized subscripts summed from 1 to 4. Using this notation (Barnett and Lothe, 1975), Eqs. (1) and (2) can be written as

$$\sum_{iJ} = E_{iJMn} \cdot Z_{Mn} \quad (7)$$

where $\sum_{iJ}, E_{iJMn}$, and Z_{Mn} are, respectively,

$$\sum_{iJ} = \begin{cases} \sigma_{ij} & J = 1,2,3 \\ D_i & J = 4 \end{cases} \quad (8)$$

$$E_{iJMn} = \begin{cases} C_{ijmn}\, J, & M = 1,2,3 \\ e_{nij} & J = 1,2,3;\ M = 4 \\ e_{imn} & J = 4;\ M = 1,2,3 \\ \kappa_{in} & J = 4;\ M = 4 \end{cases} \quad (9)$$

$$Z_{Mn} = \begin{cases} \varepsilon_{mn} & M = 1,2,3 \\ -E_n & M = 4 \end{cases} \quad (10)$$

$$U_M = \begin{cases} u_m & M = 1,2,3 \\ \phi & M = 4 \end{cases} \quad (11)$$

Let us consider a piezoelectric inclusion problem in which a region Ω in an infinite domain $\mathbb{R}^3$ has a constant transformation strain-electric field Z^*, which is free of both stress and electric displacement (Fig. 1). Mathematically, the problem is defined by the following partial differential equation (see Mikata, 2000, 2001):

$$\sum_{iJ,i} = 0 \quad (12)$$

$$\sum_{iJ} = E_{iJMn}[Z_{Mn} - Z^*_{Mn}(x)] \quad (13)$$

$$E_{iJMn}Z_{Mn} = E_{iJMn}U_{M,ni} \quad (14)$$

where the transformation strain-electric field $Z^*_{Mn}(x)$ is given by

$$Z^*_{Mn}(x) = \begin{cases} Z^*_{Mn} & x \in \Omega \\ 0 & x \in \mathbb{R}^3 - \Omega \end{cases} \quad (15)$$

Substituting (13) and (14) into (12), one obtains (see Mikata, 2000, 2001)

$$E_{iJMn}U_{M,ni} = E_{iJMn}\partial_i Z^*_{Mn}(x) \quad (16)$$

where ∂_i denotes partial differentiation with respect to x_i. It is evident from (16) that $E_{iJMn}\partial_i Z^*_{Mn}(x)$ acts as a body force-electric charge density. When the shape of the inclusion Ω is an ellipsoid, which is the most interesting case, the strain and electric field Z in Ω that result from Z^* can sometimes be determined explicitly by

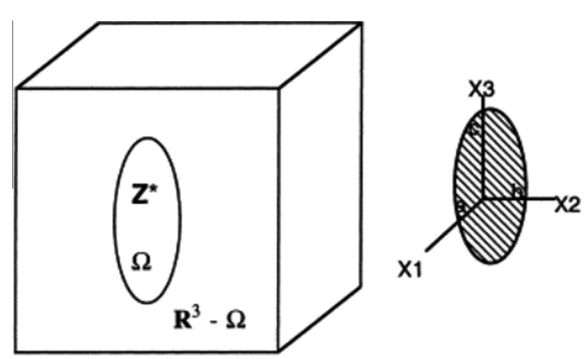

Fig. 1. Scheme of transformation strain-electric field Z^* in a region Ω in an infinite piezoelectric medium.