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Degenerate scales for boundary value problems in anisotropic elasticity

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ABSTRACT

Degenerate scales usually refer to a size effect which causes non-unique solutions of boundary integral equations for certain type of boundary value problems with a unique solution. They are closely connected to the presence of a logarithmic function in the integral kernel of the single-layer potential operator. The equations of the elasticity theory provide one of the known application fields where degenerate scales appear. The paper discusses conditions and formula for controlling and detection of the degenerate scales in the case of fully anisotropic analysis. No restrictions are considered for the material, only the loading should cause two-dimensional deformation of the anisotropic body. A technique for the evaluation of the degenerate scales is discussed and tested. The examples provide results of special simple cases and demonstrate suitability of the proposed technique in relation to calculation of degenerate scales by numerical solution of pertinent boundary integral equation by the boundary element method.

1. Introduction

Degenerate scales appear in the solution of some boundary integral equations (BIE). They are provoked in situations affected by the size of the domain, where the BIE has either multiple solutions or does not have a solution at all, while the pertinent boundary value problem (BVP) is uniquely solvable. They arise in the solution of Dirichlet BVP (DBVP) by means of a single-layer potential operator, whose weakly singular integral kernel includes a logarithmic function. The logarithmic character of the kernel is well known in isotropic plane elasticity but it is also retained in the case of anisotropy. In numerical calculations, the degenerate scales may affect the solution of DBVP, if e.g. the boundary element method (BEM) is implemented for pertinent BIE. The above phenomenon represents a well known difficulty appearing in applications of BIEs to the solution of other plane elliptic DBVPs. In potential theory, the degenerate scale for a boundary is characterized by the unit value of the logarithmic capacity of this boundary, see Jaswon and Symm (1977), McLean (2000) and Yan and Sloan (1988). Some approaches for avoiding the non-invertibility of the BIE obtained from the harmonic single-layer potential operator were studied in Chen et al. (2014, 2002b) and Christiansen (1982, 1985). A special attention to the exterior DBVP, especially for the case of domains with several holes, was paid recently in Chen et al. (2009b) and Corfdir and Bonnet (2013). Some special cases of boundary contours were discussed in Chen et al. (2005, 2009c) and Kuo et al. (2013). The degenerate scales for the biharmonic single-layer potential were analyzed in depth in Christiansen (1998, 2001) and Costabel and Dauge (1996). Under special conditions, the degenerate scales arise also in solving the Helmholtz equation (Kress and Spassov, 1983) or the Stokes equation (Dijkstra and Mattheij, 2008) by BIE.

The degenerate scales in plane isotropic elasticity were determined in numerous analytical and numerical approaches for simple circular, elliptic or annular domains in Chen et al. (2002a), He et al. (1996), Heise (1978, 1987), Vodička and Mantič (2008) and for more general domains in Kuhn et al. (1987) and Vodička and Mantič (2004a). The exterior DBVP was analyzed in Chen and Lin (2008) and Chen et al. (2009d), some examples with asymptotic behavior of degenerate scales was mentioned in Chen (2011) and Vodička and Mantič (2004b) and proved in Vodička (2013). A mathematical proof of the existence of degenerate scales was given in Constanda (1994) and Vodička and Mantič (2004b) and the upper bounds for degenerate scales were proved in Corfdir and Bonnet (2014). Theoretically well based approaches of removing the non-uniqueness from the solution of the single-layer potential BIE were proposed in Constanda (1995) and Hsiao and Wendland (1985).

So far, up to the authors' knowledge, there is no mention about the degenerate scales for anisotropic media in elasticity. Nevertheless, numerous BEM application in this field, e.g. Blázquez et al. (2006), Mantič and París (1998) and Shiah and Tan (2000), may

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under some conditions give rise to this phenomenon. The difference with respect to the standard isotropic analysis is that for a general anisotropic material the inplane and antiplane deformations do not have to be uncoupled so that if for isotropic inplane elasticity there are generally two degenerate scales and for the antiplane elasticity there exists one additional degenerate scale, see e.g. Chen et al. (2009a), there are expected three degenerate scales for general anisotropic elasticity with the displacement field depending on two dimensions, referred also to as generalized plain strain state. The objective of the present paper is, first, to develop a formula calculating the degenerate scales for general anisotropic material with no a priori assumption about material symmetry and, second, to assess its validity by a numerical analysis of the problem implementing the symmetric Galerkin BEM (SGBEM) (Bonnet et al., 1998) for the solution of the pertinent BIE.

In the following part, Section 2, some basic equations are summarized. Simultaneously, basic relations from anisotropic elasticity related to the use of the single-layer potential are mentioned, based on the theory developed in Ting (1996) and Hwu (2010). In the introduction to Section 3, known relations for finding degenerate scales are recalled from Vodička and Mantič (2004b, 2008), the particular relations for anisotropic materials are described in the subsections. Namely, an estimate which guarantees the invertibility of the single-layer potential operator is given in Section 3.1 and derivation of the formula for calculating the degenerate scales is provided in Section 3.2. The formula is tested in the section of examples, Section 4, by an SGBEM code and in one problem also by a comparison with an analytical solution. The paper also includes two appendices which provide a calculation of an integral (Appendix A) required in Section 3.1 and a brief summary of anisotropic fundamental solution (Appendix B) based on the Stroh formalism.

2. Solution of a Dirichlet boundary value problem by the single-layer potential

Let us consider an elastic body, a domain $\Omega \times \langle -h; h\rangle \subset \mathbb{R}^3$, $\Omega \subset \mathbb{R}^2$ with a bounded Lipschitz boundary (McLean, 2000) $\partial\Omega = \Gamma$ as shown in Fig. 1.

Consider a fixed cartesian coordinate system x_i $(i = 1, 2, 3)$ placed so that Ω resides in the coordinate plane x_1x_2. Let $\mathbf{u} = (u_1, u_2, u_3)^\top$ be the displacement solution of the following Dirichlet problem for the Navier equation in the case of two-dimensional deformations of the body (or the generalized plane strain state) introduced such that no displacement variable depends on the coordinate x_3:

$$c_{ijkl}u_{k,lj}(x) = 0, \quad x \in \Omega, \tag{1a}$$

$$u_i(x) = g_i(x), \quad x \in \Gamma, \tag{1b}$$

where we denoted $x = (x_1, x_2)$ and we used the positive definite fourth-order tensor of elastic stiffnesses c_{ijkl} (Gurtin, 1972) with no special consideration of material symmetry which is then generally anisotropic. Let us stress that even though the antiplane displacements u_3 are allowed, all deformations depend only on inplane coordinates x_1 and x_2. It also means that l and j could be summed only for 1 and 2, cf. also (4b) below for used elastic parameters.

Let $\mathbf{U}(x,y) = \left(U_{ij}(x,y)\right)_{i,j=1,2,3}$ be the symmetric second-order tensor of the fundamental solution of the Navier equation (1a), i.e. displacements at the space point x due to unit forces $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ applied at the point $y = (y_1, y_2)$. In fact the forces are line forces applied along x_3 axis as long as the loading does not depend on x_3 when the generalized plane strain state is to be considered. The fundamental solution $\mathbf{U}$ is given according to Ting (1996) as:

$$\begin{aligned}\mathbf{U}(x,y) &= \frac{1}{2\pi}\left[-\mathbf{H}\ln\frac{r}{r_0} - \pi\left(\mathbf{H}\mathbf{S}^\top(\Theta) + \mathbf{S}\mathbf{H}(\Theta)\right) + \mathbf{K}\right] \\ &= -\frac{1}{2\pi}\mathbf{H}\ln\frac{r}{r_0} - \mathbf{Z}(\Theta) + \frac{1}{2\pi}\mathbf{K},\end{aligned} \tag{2}$$

where the polar coordinates $y_1 = x_1 + r\cos\Theta, y_2 = x_2 + r\sin\Theta$ are used, r_0 is an arbitrary constant to make the argument of logarithm dimensionless (it is usually set to a unit value), and $\mathbf{K}$ is an arbitrary constant symmetric matrix.

The material characteristics usually defined by the elastic stiffness tensor c_{ijkl} are included here in the form of the Barnett–Lothe tensors $\mathbf{H}$ and $\mathbf{S}$. All matrices in Eq. (2) can be obtained by known formulae of anisotropic elasticity as follows (see also Ting, 1996):

$$\begin{pmatrix}\mathbf{S}(\Theta) & \mathbf{H}(\Theta)\\ -\mathbf{L}(\Theta) & \mathbf{S}^\top(\Theta)\end{pmatrix} = \frac{1}{\pi}\int_0^\Theta \begin{pmatrix}-\mathbf{T}^{-1}(\theta)\mathbf{R}^\top(\theta) & \mathbf{T}^{-1}(\theta)\\ -\mathbf{Q}(\theta) + \mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\mathbf{R}^\top(\theta) & -\mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\end{pmatrix}\mathrm{d}\theta, \tag{3a}$$

$$\begin{pmatrix}\mathbf{S} & \mathbf{H}\\ -\mathbf{L} & \mathbf{S}^\top\end{pmatrix} = \begin{pmatrix}\mathbf{S}(\pi) & \mathbf{H}(\pi)\\ -\mathbf{L}(\pi) & \mathbf{S}^\top(\pi)\end{pmatrix}, \tag{3b}$$

with

$$\begin{pmatrix}\mathbf{Q}(\Theta) & \mathbf{R}(\Theta)\\ \mathbf{R}^\top(\Theta) & \mathbf{T}(\Theta)\end{pmatrix} = \begin{pmatrix}\mathbf{I}\cos\Theta & \mathbf{I}\sin\Theta\\ -\mathbf{I}\sin\Theta & \mathbf{I}\cos\Theta\end{pmatrix}\begin{pmatrix}\mathbf{Q} & \mathbf{R}\\ \mathbf{R}^\top & \mathbf{T}\end{pmatrix}\begin{pmatrix}\mathbf{I}\cos\Theta & -\mathbf{I}\sin\Theta\\ \mathbf{I}\sin\Theta & \mathbf{I}\cos\Theta\end{pmatrix}. \tag{4a}$$

The relation of the introduced matrices to the elastic stiffnesses c_{ijkl} is provided by the relations

$$T_{ik} = c_{i2k2}, \quad R_{ik} = c_{i1k2}, \quad Q_{ik} = c_{i1k1}. \tag{4b}$$

Another way of the matrices definition is presented in Appendix B where also some of their useful properties are mentioned.

The solution $\mathbf{u}$ of DBVP (1) in the indirect method can be expressed in the form of the single-layer potential

$$\mathbf{u}(x) = \int_\Gamma \mathbf{U}(x,y)\boldsymbol{\varphi}(y)\mathrm{d}\Gamma(y) = \mathcal{U}\boldsymbol{\varphi}(x), \tag{5}$$

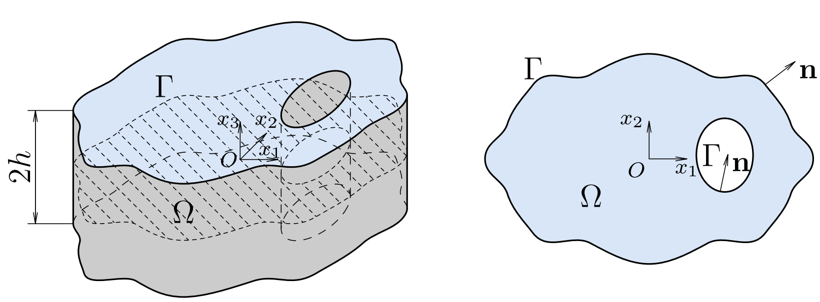

Fig. 1. Description of a cylindrical bounded domain and its cross-section.