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Coexisting equilibria and stability of a shallow arch: Unilateral displacement-control experiments and theory

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ABSTRACT

The equilibria and stability of a shallow prestressed arch (beam–column) are investigated theoretically and experimentally. The deflection of the arch is unilaterally constrained by a displacement-control device. Both snap-through and remote coexisting equilibria are observed. Force–deflection curves for primary and secondary equilibrium branches are measured for varying constraint locations. The effect of the constraint location on the critical condition at which stability is lost, resulting in a jump to a remote equilibrium, is investigated. Good agreement is attained between experimental data and theoretical results (based on minimization of the constrained strain energy and an inextensibility assumption).

1. Introduction

The shallow arch has been used extensively to illustrate the non-linear structural behavior commonly found in more complex civil, mechanical, and aerospace structures. Unlike its linear counterpart, the flat beam, the arch can exhibit complex responses when transversely loaded, for example *snap-through* buckling in which the arch suddenly jumps from one equilibrium configuration to a remote coexisting configuration (Fung and Kaplan, 1952). This remote configuration is often associated with an inverted position, but there can be multiple competing configurations to which the arch can jump (Pi and Bradford, 2014). The purpose of the present work is to characterize these remote equilibria and their stability.

Previous work has primarily focused on a *dead loading* formulation, motivated by the classical testing procedure which involves hanging masses from the arch. This is equivalent to treating the applied force as the control parameter (Fung and Kaplan, 1952; Roorda, 1965). In the present study, an actuating device constrains the transverse deflection of the arch at the point of application (i.e. *displacement control*) (Chen and Hung, 2011), and the reaction force is measured via a collocated load cell. The device is not rigidly connected to the arch and therefore is capably of *pushing* but not pulling—the force measured by the load cell is greater than or equal to zero. As such, the deflection of the arch (at the point of contact) must adhere to a unilateral (or inequality) constraint prescribed by the displacement of the device (Mirasso and Godoy, 1997; Godoy and Mirasso, 2003). Unilateral constraints in buckling problems are not without precedence, e.g., discrete systems (Burgess, 1971; Klarbring, 1988), beams (Adan et al., 1994; Domokos et al., 1997; Villaggio, 1979; Tzaros and Mistakidis, 2011; Sun and Natori, 1996; Hexiang et al., 1999; Chen and Ro, 2010), and plates (Chai, 2002). Recently, Lu and Lu (2014) studied the behavior of a shallow arch constrained by a *fixed* rigid plate. The post-buckled response of bilaterally constrained columns has been investigated (Chai, 1998). Constrained buckling has more general applications in three dimensional structures, such as stents (McGrath et al., 2014), deep drilling (Thompson et al., 2012), and DNA structures (Hirsh, 2013).

Under dead loading, snap-through occurs when stiffness is lost at a horizontal tangency on the force–deflection curve (a limit point) or at a bifurcation point. A greater portion of the force–deflection curve is stabilized under displacement control. For a unilateral constraint, snap-through occurs when the reaction goes to zero and contact is lost. For a bilateral constraint (i.e. the actuating device is rigidly connected to the arch (Camescasse et al., 2014)), negative reaction forces can be achieved and snap-through does not occur. In all cases, a single branch of the force–deflection curve is explored when the control parameter (load or displacement) quasi-statically changes. Such natural loading histories preclude

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the exploration of remote equilibria. The force–deflection curve can exhibit "looping" with multiple coexisting branches (Sabir and Lock, 1973). Harrison (1978) demonstrated these looping branches for a shallow circular arch under central or near central dead loading using a discrete element method. The first objective of this paper is to experimentally explore these secondary equilibria and assess their stability. The measured data is compared to an experimentally-motivated theoretical formulation for displacement-control tests. The theory involves minimizing the strain energy subject to an inequality (unilateral) constraint and an equality constraint to account for inextensibility in the arch.

Much of the earliest work on the buckling of shallow arches focused on symmetrically loaded arches (Lock, 1966; Hsu et al., 1969; Fung and Kaplan, 1952; Chen et al., 2009; Moghaddasie and Stanciulescu, 2013; Bradford et al., 2002) , either uniformly distributed loads (UDLs) or central point loads. However, the critical buckling load is sensitive to small imperfections in the load location. Harrison (1982) numerically demonstrated that the intensity of load that an arch can support is appreciably reduced when a UDL does not cover the whole span. A classic saddle-node bifurcation is observed when a central point load is slightly offset (Harrison, 1978). Roorda (1965) experimentally showed this cusp imperfection-sensitivity, and Thompson and Hunt (1983) subsequently found analytical expressions for the cusp. The critical buckling load is also sensitive to the thermal environment (Virgin et al., 2014) and slight geometric imperfections (Schreyer, 1972; Roorda, 1968; Pi and Bradford, 2012), which are outside the scope of the present work.

Hsu (1968) and Plaut (1979) analytically assessed the influence of the load position—not restricted to the vicinity of the arch center, but along the entire span—on the critical buckling load. Plaut showed that, for a large enough rise, a bifurcation (cusp) occurs at the midpoint while two minima move outward from it. The present work experimentally validates the predicted influence of load position on critical buckling load (Plaut, 1979). Additionally, this paper extends these results, both analytically and experimentally, to the critical mode-jumping load on a secondary equilibrium branch, which to date has never been shown.

2. The shallow arch

2.1. The experimental setup

A photograph of the system is shown in Fig. 1(a). The arch was made of a flat relatively-stiff quasi-isotropic carbon fiber bar with the following cross-sectional dimensions: width 25.4 mm and thickness 0.794 mm. The bending stiffness of the bar was experimentally determined to be $EI = 48.8$ kN mm^2.

The bar was pinned at both ends. The pin supports (see Fig. 1(b) and (c)) consisted of 12.7 mm rods mounted in bearing fixtures to permit pure rotation with negligible friction. The rods were machined to permit the bar to lie approximately on the rods' axes of rotation, which were fastened with a backing plate via two threaded holes. Initially, the bar was installed in a flat configuration, spanning a length of $L = 292.1$ mm. Then, by applying an end displacement, the strut buckles into its fundamental buckled shape, depicted in Fig. 1(b).

The arch was laterally actuated at a single point via a screw device whose translational motion was transferred through a linear bearing. At the interface between the linear bearing and the arch, a load cell measured the normal force at the point of contact. The horizontal point of contact was varied by sliding the apparatus along guide rails. The deflection at a point on the arch was measured by a laser proximity sensor. During the quasi-static tests, the force and deflection measurements were simultaneously acquired.

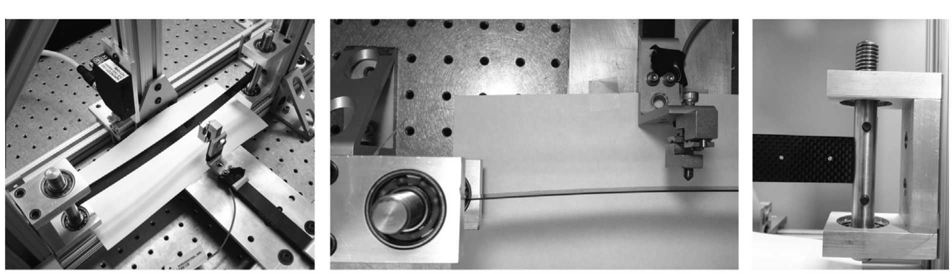
2.2. Testing procedure

Snap-through in shallow arches has classically been studied under dead loading (Roorda, 1965). In the present experiments, an adjustable bound on the lateral deflection is enforced by the screw device, commonly referred to as *displacement control*. Displacement control stabilizes a greater portion of the force–deflection curve (Virgin et al., 2014), but only positive reaction forces are permitted, in our case, because the load cell is not attached to the arch: i.e., the load cell pushes but does not pull.

Two objectives of this study are: (1) analyze the influence of constraint position on the equilibria paths, and (2) investigate the existence of secondary equilibria and their stability. To investigate the former, various combinations of constraint position and deflection measurement position are tested; to investigate the latter, the system is manually perturbed to a secondary, stable equilibrium path. The testing procedure used for a single constraint application position and deflection measurement position is broken down into three parts: the *primary*, *secondary*, and *tertiary* branches. For reference, a representative force–deflection curve is given in Fig. 2 for a reaction force measured left-of-center and the deflection measured right-of-center. The testing procedure is as follows:

Primary branch:

1. The load cell is initially not in contact with the arch, and the reaction force is zero ($F = 0$). The arch takes its fundamental deformed shape (i_0), a half sinusoid.
2. The load cell is advanced toward the center of curvature, makes contact with the arch, and gradually constrains the deflection at the point of application. Due to the load offset, the arch takes the deformed shape (i), a combination of shape (i_0) and a full sinusoid. The force initially increases, reaches a maximum value (this is the loss of stability under dead loading—a classic saddle-node bifurcation), and then decreases.



(a) (b) (c)

Fig. 1. (a) The experimental test system, (b) the pin support at one end and the load cell attached to a displacement-control screw device, and (c) the pin support.