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# An arbitrary piezoelectric inclusion with weakly and highly conducting imperfect interface

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ABSTRACT

In the current study, we rigorously analyze an arbitrarily shaped piezoelectric inclusion surrounded by an infinite isotropic piezoelectric matrix subject to antiplane shear and in plane electric field loadings. The inclusion and matrix are separated by a homogeneously imperfect interface that characterizes a spring type interaction between the elastic and electric interfacial boundary conditions. Furthermore, the boundary conditions for a mechanically compliant, weakly conducting and mechanically compliant, highly conducting interface are incorporated into the analysis. Using complex variable techniques the potential function inside the inclusion is formulated as a Faber Series expansion and a system of linear algebraic equations for a closed form solution is developed for the corresponding Faber coefficients under a finite number of terms. Under this approach, expressions for both the elastic and electric fields are developed for the inclusion and matrix. The results are presented in exact form for an elliptic inclusion and numerically simulated for a finite number of terms for purposes of verification. Additionally, the cases of a square and star inclusion geometry are analyzed and results are presented numerically. The results clearly demonstrate that not only is the stress distribution inside the inclusion interface non-uniform, but that the magnitude of the peak stresses are highly dependent on the inclusion shape and imperfect interface condition.



## 1. Introduction

In the modern world of structures, machines and computers, composite materials have a well deserved niche. Piezoelectric composites can be found in a multitude of electrical and electromechanical systems and as such, the problem of a piezoelectric inclusion has received a considerable amount attention in recent years (see, for Example Wang and Sudak, 2007; Gao and Noda, 2004; Pan, 2004; Shen and Hung, 2012; Shen et al., 2010; Sudak, 2003; Tiersten, 1969).

Mishan and Qijun studied the plane problem of an elliptical piezoelectric inclusion (Mishan and Qijun, 1998). Their work employed the use of the Faber Series technique and the analysis was conducted on an inclusion/matrix of similar material with a perfect interface. Gao and Noda analyzed the problem of an arbitrary piezoelectric inclusion in an antiplane setting (Gao and Noda, 2004). They also incorporated the Faber Series technique in their solution scheme and studied a wide range of inclusion geometries ranging from an ellipse to a square, incorporating elastic mismatch in their analysis. Wang and Sudak presented the case of a piezoelectric bimaterial with an imperfect interface and near by screw dislocation (Wang and Sudak, 2007). This work established a generalized method for formulating both weakly and highly conducting electric imperfect interface conditions under a similar guise for linear piezoelectric materials in antiplane elasticity. Shen et al. formulated a generalized approach for three phase piezoelectric inclusions of arbitrary shape (Shen et al., 2010). In their work Shen et al. analyzed square, rectangular, and triangular inclusion geometries with dissimilar inclusion/interphase/matrix materials through use of the Faber Series technique under the context of perfect bonding. Shen and Hung extended the piezoelectric inclusion case to incorporate magnetoelectroelastics in their analysis of an arbitrary shaped inclusion (Shen and Hung, 2012). They too employ the use of the Faber Series technique with dissimilar inclusion/matrix material properties analyzing perfectly bonded triangular and star shaped inclusion geometries.

The aforementioned works generally analyze piezoelectric inclusions with at least one of the following simplifications; perfect bonding along the interface or a simple inclusion boundary geometry (such as a circle or an ellipse). In reality, these assumptions do not adequately describe the interaction between the inclusion and

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matrix across the boundary between them because they cannot account for phenomena such as interfacial damage and debonding. This void has been partially filled through the works of Gao and Noda (2004), Ru (2003), Shen and Hung (2012), Sudak and Wang (2013) and Shen et al. (2010) where analytical models of imperfectly bonded elastic inclusions and piezoelectric bimaterials have been developed. Of specific interest are the models incorporating the use of the Faber Series technique; which allows for the introduction of an arbitrary inclusion geometry where traditional conformal mapping techniques fall short due to the immense complexity of the mapping function for geometries that are non-elliptic and the lack of an ability to simultaneously map both the interior and exterior of a simply connected domain. The Faber Series technique circumvents these difficulties through the construction of the inclusion potential function as a Faber Series (Gao and Noda, 2004; Shen and Hung, 2012; Sudak and Wang, 2013; Shen et al., 2010).

The objective of the present work is to rigorously model the interaction between an piezoelectric inclusion embedded in an piezoelectric matrix separated by a arbitrary shaped homogeneously imperfect interface. The interface will first be assumed to operate as a mechanically compliant, dielectrically weakly conducting interface and then later as a mechanically compliant, dielectrically highly conducting interface. The generally accepted linear spring type model will be used for the imperfect interface with continuous tractions and discontinuous displacements (Wang and Sudak, 2007; Sudak and Wang, 2013). In this model the traction components exhibit a proportionality through the spring type parameters to their respective displacements where, given a thin interphase layer separating the inclusion from the surrounding matrix, the idea of an imperfect interface model is to simulate the thin interphase layer as a 2-dimensional curve of vanishing thickness '*t*' where the material properties of the interphase layer are represented by the so-called 'spring-factor' type interface parameters. As an example, for the mechanically compliant weakly conducting case, the mechanical interface parameter represents the ratio of the elastic stiffness and the interphase thickness '*t*' and the electric interface parameter represents the product of the interphase conductivity and thickness '*t*' (Benveniste and Miloh, 2001; Chen, 2001). The dielectrically weakly conducting interface experiences continuity across the interface with respect to the normal electric displacement and discontinuity for the electric potential whereas the dielectrically highly conducting interface experiences continuity across the interface with respect to the tangential electric field and a discontinuity in the normal electric displacement. In a similar fashion to the mechanical properties of the interface, the jump in the electrical potentials/displacements is proportional to the normal electric displacements/tangential field across the interface for the weakly conducting interface and the highly conducting interface respectively.

Section 2 aims to discuss the foundations of the piezoelectric constitutive model and the introduction of the familiar Laplace equation from said model. Employing a complex variable formulation, in which the Laplace equation is solved via complex potential functions, the boundary value problem for the mechanically compliant dielectrically weakly conducting interface between the inclusion and matrix is defined. Section 3.2, through the introduction of the Faber Series for the potential function inside the inclusion in conjunction with the previously established boundary conditions, shows the derivation of the complex potential function for the matrix through the process of analytic continuation. Following Sections 3.2 and 3.3 outlines the incorporation of the imperfect interface condition and outlines the process to obtaining a generalized solution for arbitrary inclusion geometries. In Section 4 the case of a mechanically compliant, highly conducting interface is developed and a second generalized solution is obtained. In Section 5 the cases of elliptical, triangular, square, and star shaped inclusions are examined and analytical solutions for the stress and electric displacement/field around the inclusion-matrix interface are provided. Finally in Section 6 the results of the study are discussed and some conclusions are made.

## 2. Mathematical preliminaries

In this work, the case of a piezoelectric inclusion will be modeled by considering a semi-infinite domain in $\mathcal{R}^2$ that is simply connected containing a single arbitrarily shaped inclusion with material properties distinct from the surrounding matrix. The complex coordinate $z = x_1 + ix_2$ represents a singular point $(x_1, x_2)$ in $\mathcal{R}^2$. The boundary curve of the inclusion matrix interface will be referred to as $\partial L$ and the regions of the inclusion and matrix will be referred to as $D_1$ and $D_2$ respectively (see Fig. 1). As such, any future subscripts and superscripts taking the value of 1 or 2 will be understood to refer to the inclusion and matrix respectively. It is assumed that both the inclusion and matrix are isotropic with respect to the $x_1x_2$ plane and possess the same poling direction; the positive $x_3$ axis. The inclusion and matrix will be exposed to remote anti-plane shear and in-plane electric field loadings and in addition to the prescribed loading, the resulting anti-plane deformation in the far field of the matrix shall satisfy simple shear conditions.

### 2.1. Fundamental equations of piezoelectric materials

Linear piezoelectric materials, operating in a fixed cartesian coordinate system, are governed by the following constitutive equations (Tiersten, 1969) in antiplane elasticity

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0, \tag{1}$$

$$\zeta_{3i} = \frac{1}{2}(u_{3,i} + u_{i,3}), \quad E_i = -\varphi_{,i}, \tag{2}$$

$$\sigma_{3i} = C_{44}2\zeta_{3i} - e_{15}E_i, \quad D_i = e_{15}2\zeta_{3i} + \epsilon_{11}E_i \quad i = 1, 2. \tag{3}$$

In the above equations a comma represents the convention of differentiation and repeated indices represent summations following the traditional Einstein summation convention. $C_{44}, e_{15}$, and $\epsilon_{11}$ are material constants corresponding to the elastic, piezoelectric, and dielectric permitivity of a piezoelectric material, respectively. Additionally, $\sigma_{3i}, D_i, E_i, \zeta_{3i}, u_i$, and $\varphi_{,i}$ for $i = 1, 2$ are the stress,

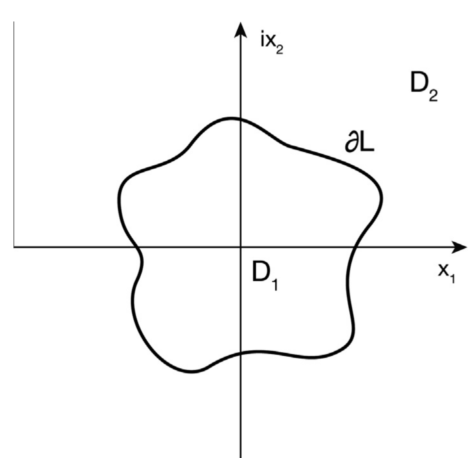


**Fig. 1.** Piezoelectric inclusion ($D_1$) bounded by curve $\partial L$ embedded in a piezoelectric matrix ($D_2$).