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The effective elastic moduli of columnar composites made of cylindrically anisotropic phases with rough interfaces

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ABSTRACT

The present work aims to determine the effective elastic moduli of a composite having a columnar microstructure and made of two cylindrically anisotropic phases perfectly bonded at their interface oscillating quickly and periodically along the circular circumferential direction. To achieve this objective, a two-scale homogenization method is elaborated. First, the micro-to-meso upscaling is carried out by applying an asymptotic analysis, and the zone in which the interface oscillates is correspondingly homogenized as an equivalent interphase whose elastic properties are analytically and exactly determined. Second, the meso-to-macro upscaling is accomplished by using the composite cylinder assemblage model, and closed-form solutions are derived for the effective elastic moduli of the composite. Two important cases in which rough interfaces exhibit comb and saw-tooth profiles are studied in detail. The analytical results given by the two-scale homogenization procedure are shown to agree well with the numerical ones provided by the finite element method and to verify the universal relations existing between the effective elastic moduli of a two-phase columnar composite.

1. Introduction

An important class of composites, referred to as columnar composites, have the microstructure such that their properties are homogeneous along one direction but heterogeneous in its transverse plane. Within the class of columnar composites fall, for example, porous media with parallel cylindrical pores, fibrous composites consisting of a homogeneous matrix reinforced by aligned parallel continuous homogeneous fibers, or polycrystalline aggregates formed of columnar monocrystals. The microstructure of columnar composite is much more complicated than the one of layered composites but simpler than that of particulate composites.

Due to the practical importance and technological interest of columnar composites, a great number of micromechanical models have been proposed to determine the effective mechanical properties of them and, in particular, those of fiber-reinforced composites. These models include diluted, self-consistent, Mori–Tanaka, differential, and Ponte Castaneda–Willis estimation schemes. They equally comprise the generalized self-consistent model (GSCM) initiated by Van der Poel (1958), improved by Smith (1974, 1975) and completed by Christensen and Lo (1979). It is also very useful to mention the composite cylinder assemblage (CCA) model introduced first by Hashin and Rosen (1964) and recast then by Hashin (1972, 1979) on the basis of the concept of neutral inclusions. All these micromechanical models providing closed-form estimates or exact results for the effective properties of columnar composites require using Eshelby's solution for a cylindrical inclusion of circular or elliptic cross-section in an infinite homogeneous medium (Eshelby, 1957; Eshelby, 1961) or the solution for a cylindrical inhomogeneity of circular cross-section coated concentrically by a circular cylindrical shell. For this reason, once the interface between two phases of a columnar composite can no longer be considered as smooth but rough, none of the aforementioned micromechanical models is still valid.

In a variety of situations and for different reasons, the consideration of rough surfaces and interfaces is unavoidable. More fundamentally, a surface or interface which is smooth at a given scale turns out often to be rough at a smaller scale. In the physics and mechanics of solids or fluids, a great number of studies have been dedicated to rough surfaces and interfaces (see, e.g., Zaki and

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Neureuther, 1971; Waterman, 1975; Cheng and Olhoff, 1981; Talbot et al., 1990; Belyaev et al., 1992, 1998; Abboud and Ammari, 1996; Achdou et al., 1998; Bao and Bonnetier, 2001; Singh and Tomar, 2007, 2008). Two approaches have been proposed for the investigation of rough interfaces. When the amplitude of the roughness of a rough interface is much smaller than its wavelength (or period), the approach based on an appropriate perturbation technique is employed (see e.g., Sato et al., 2012 and the relevant references cited therein). If the wavelength is much smaller than the amplitude, the homogenization approach is justified and efficient (see, e.g., Kohler et al., 1981; Brizzi, 1994).

The present work addresses the problem of determining the effective elastic moduli of a columnar composite made of two cylindrically orthotropic phases between which the circular cylindrical interface oscillates quickly and periodically along the circular circumferential direction. To solve this problem, a two-scale homogenization method is developed. First, performing the micro-to-meso upscaling by an asymptotic analysis, the zone in which the interface undulates is homogenized as an equivalent interphase. Remarkably, the elastic properties of this interphase can be analytically and exactly determined in a compact way. Second, carrying out the meso-to-macro upscaling by using the composite cylinder assemblage model, closed-form expressions are derived for the effective elastic moduli of the composite which is assumed to be transversely isotropic at the macroscopic scale. Two rough interface configurations are investigated in details. In the first configuration, the cross-section of the rough interface presents a comb profile. The second configuration corresponds to the case where the cross-section of the rough interface exhibits a saw-tooth profile. The corresponding analytical results obtained by the two-scale homogenization method are finally compared with the numerical results provided by the finite element method and checked against the universal relations which must be verified by the effective elastic moduli of a two-phase columnar composite. The comparison and check made confirm the validity of the proposed method.

The paper is organized as follows. In Section 2, the setting of the problem under investigation is specified. In particular, the local governing equations, the description of the rough interface and the final macroscopic constitutive equation are given. Section 3 is dedicated to carrying out an asymptotic analysis so as to homogenize a rough interface zone as an equivalent interphase which may be still heterogeneous along the radial direction but becomes homogeneous on the cylindrical surface normal to the radial direction. In particular, we prove that the elastic moduli of the equivalent interphase correspond to those obtained by the homogenization of a layered composite whose layering direction coincides with the undulation one. In Section 4, the effective elastic moduli of the composite under consideration are exactly determined by using the composite cylinder assemblage model for the resulting matrix/interphase/fiber composite. In Section 5, the derived analytical results are compared with the corresponding numerical results obtained by the finite element method and checked with respect to the universal relations existing between the effective elastic moduli. Finally, a few concluding remarks are given in Section 6.

2. Local governing relations and rough interfaces

In a three-dimensional Euclidean space $\mathbb{R}^3$, consider a composite material consisting of two phases and exhibiting a columnar microstructure. The two constituent phases of the composite under consideration are assumed to be linearly elastic, cylindrically anisotropic and perfectly bonded together across their interface. More precisely, relative to the cylindrical coordinate system (r, θ, z) associated with a cylindrical orthonormal basis $\{\mathbf{e}_r,\ \mathbf{e}_\theta,\ \mathbf{e}_z\}$ with the unit vector $\mathbf{e}_z$ orientated along the cylinder axis, the components σ_{ij} of the Cauchy stress tensor $\boldsymbol{\sigma}$ are related to the components ε_{ij} of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ by

$$\sigma_{ij} = \mathcal{L}_{ijpq}\varepsilon_{pq}, \tag{1}$$

where $\mathcal{L}_{ijpq}$ are the components of the fourth-order elastic stiffness tensor $\mathbb{L}$ and have the usual symmetries $\mathcal{L}_{ijpq} = \mathcal{L}_{jipq} = \mathcal{L}_{pqij}$. By hypothesis, the phases are cylindrically anisotropic, so that all the elastic moduli $\mathcal{L}_{ijpq}$ are independent of the cylindrical coordinates (r,θ,z). In other words, the phases are cylindrically homogeneous but heterogeneous with respect to the Cartesian coordinates (x,y,z) associated with an orthonormal basis $\{\mathbf{e}_x,\ \mathbf{e}_y,\ \mathbf{e}_z\}$.

In the cylindrical coordinate system, using the two-to-one subscript identification $rr\equiv 1,\ \theta\theta\equiv 2,\ zz\equiv 3,\ \theta z\equiv 4,\ zr\equiv 5$ et $r\theta\equiv 6$, the cylindrically anisotropic elastic law (1) can be written in the following matrix form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sqrt{2}\sigma_4 \\ \sqrt{2}\sigma_5 \\ \sqrt{2}\sigma_6 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} & 0 & 0 & 0 \\ L_{12} & L_{22} & L_{23} & 0 & 0 & 0 \\ L_{13} & L_{23} & L_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2L_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix}. \tag{2}$$

In this work, we are interested in the case where each phase belongs to one of the isotropic, cubic, transversely isotropic, tetragonal and orthotropic symmetric classes. These five symmetry classes under consideration are now specified as follows: (i) if $L_{11},\ L_{22},\ L_{33},\ L_{12},\ L_{13},\ L_{23},\ L_{44},\ L_{55}$ and L_{66} are independent, the constitutive law (2) is cylindrically orthotropic; (ii) when $L_{11},\ L_{33},\ L_{12},\ L_{13},\ L_{44}$ and L_{66} are independent, $L_{11}=L_{22},\ L_{13}=L_{23}$ and $L_{44}=L_{55}$, the relation (2) becomes cylindrically tetragonal; (iii) in the case where $L_{11},\ L_{33},\ L_{12},\ L_{13}$ and L_{44} are independent, $L_{11}=L_{22},\ L_{13}=L_{23},\ L_{44}=L_{55}$ and $L_{66}=\frac{1}{2}(L_{11}-L_{12})$, Eq. (2) is of cylindrically transverse isotropy; (iv) if $L_{11},\ L_{12}$ and L_{44} are independent but $L_{11}=L_{22}=L_{33}$ and $L_{23}=L_{12}=L_{13},\ L_{44}=L_{55}=L_{66}$, the material described by (2) is cylindrically cubic; (v) lastly, when L_{11} and L_{12} are independent, $L_{11}=L_{22}=L_{33},\ L_{23}=L_{12}=L_{13},\ L_{44}=L_{55}=L_{66}=\frac{1}{2}(L_{11}-L_{12})$, then the material characterized by (2) is isotropic.

In the cylindrical coordinates, the components of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ are related to the displacement field $\mathbf{u}=(u_r,u_\theta,u_z)$ by

$$\varepsilon_1 = u_{r,r},\quad \varepsilon_2 = \frac{1}{r}(u_{\theta,\theta}+u_r),\quad \varepsilon_3 = u_{z,z},\quad \varepsilon_4 = \frac{1}{2}\left(u_{\theta,z}+\frac{1}{r}u_{z,\theta}\right),$$
$$\varepsilon_5 = \frac{1}{2}(u_{r,z}+u_{z,r}),\quad \varepsilon_6 = \frac{1}{2}\left(u_{\theta,r}+\frac{1}{r}u_{r,\theta}-\frac{1}{r}u_\theta\right). \tag{3}$$

The Cauchy stress tensor field $\boldsymbol{\sigma}$ must verify the following motion equations:

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{b} = \rho\ddot{\mathbf{u}} \tag{4}$$

where $\mathbf{b}=(b_r,b_\theta,b_z)^T$ is the body force vector, ρ is the volumic mass density and a superposed dot denotes the differentiation with respect to the time t.

Next, by substituting (1) into (4) and by taking into account (3), the motion equation (4) can be rewritten in the following compact form

$$\left(\mathbf{C}^{(pk)}\mathbf{u}_{,k}\right)_{,p} + \frac{1}{r}\mathbf{C}^{(1k)}\mathbf{u}_{,k} + \mathbf{b} = \rho\ddot{\mathbf{u}}, \tag{5}$$

where

$$(\bullet)_{,1} = (\bullet)_{,r},\quad (\bullet)_{,2} = \frac{1}{r}(\bullet)_{,\theta},\quad (\bullet)_{,3} = (\bullet)_{,z} \tag{6}$$