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Poynting and axial force–twist effects in nonlinear elastic mono- and bi-layered cylinders: Torsion, axial and combined loadings

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ABSTRACT

The Poynting effect, in which a cylinder elongates or contracts axially under torsion, is an important nonlinear phenomenon in soft materials. In this paper, analytical solutions are obtained for homogeneous and bi-layered cylinders under torsion, axial and combined loadings, employing second-order elasticity and Lagrangian equilibrium equations. Explicit parameters for judging the sign of the Poynting effect are given. It is found that the effect in a soft composite may be significantly amplified over that in homogeneous materials and that it is strongly influenced by the interface position and by the material configuration in the composite. A coupled axial force–twist effect under combined loading, i.e., the twist of a torsionally loaded cylinder can be affected by the axial loading, is also found. Comparison of the predictions with the torque-tension-twist data for cardiac papillary muscles shows reasonable agreement. The solutions also provide the basis for a mechanistic method of determining third-order elastic constants.

1. Introduction

Soft materials such as synthetic polymers have a very diverse range of applications, e.g., drug delivery (Brazel and Peppas, 1999), tissue scaffolding (Stammen et al., 2001), miniature sensors (Han et al., 2002) and actuators (Beebe et al., 2000). The mechanical properties of biopolymer networks also play an essential role in many physiological functions of cells and tissues (Janmey et al., 2007). A common feature of soft materials is their nonlinearity, which distinguishes them from materials such as ceramics and metals. The Poynting effect, i.e., a cylindrical specimen elongates or contracts in the axial direction under torsion, is one of the important nonlinear phenomena. It has long been recognized that many elastic materials exhibit the positive Poynting effect when subjected to torsion (Poynting, 1909; Rivlin, 1953). However, recent discovery shows that many biopolymers exhibit the negative Poynting effect under torsion or shear. Networks of semiflexible biopolymers (actin, vimentin, neurofilaments) may generate negative normal stresses of magnitude comparable to the applied shear stress (Janmey et al., 2007; Kang et al., 2009; Conti and MacKintosh, 2009), or equivalently, they contract axially under torsion.

A number of models have been used to investigate the Poynting effect, as briefly reviewed below. Mihai and Goriely (2011) imposed a set of adscititious inequalities related to material parameters for describing the Poynting effect of hyperelastic materials subjected to simple or pure shear. Both the positive and negative effects are possible according to these inequalities. In a later paper, they also studied the Poynting effect via the finite element method (Mihai and Goriely, 2013). In this work, they emphasized that for a small triaxial stretch superimposed on simple shear, the normal stress contribution due to the triaxial stretch should be removed to judge if the Poynting effect exists. Similarly, when *pure* shear is imposed on an axially stretched solid, the Poynting effect should be defined with respect to the length after pre-stretching and not the initial length before any loading is imposed. Horgan and Murphy (2011), employing a logarithmic form for the strain energy densities of incompressible anisotropic materials, showed that the negative Poynting effect is generated in the presence of large material anisotropy and is also present under certain loading conditions. Misra et al. (2010) experimentally showed that the negative normal stress in anisotropic myocardial tissue under shear is larger than in Sylgard gel, which they also described theoretically using an exponential form and an Ogden form of the strain energy densities for the tissue and the gel, respectively. Wu and Kirchner (2010) showed that the negative Poynting effect is generally possible for biogels. Their work is based on second-order elasticity. Kang et al. (2009) developed microstructural models which

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consider filament bending and tension energies as well as thermal fluctuations. These are used to predict the direction and magnitude of the normal stress under shear. Janmey et al. (2007) developed a microstructural model, based on the transition from a bending-dominated to a stretching-dominated phase of the filaments, in order to interpret the negative normal stress effect. Zubov (2001) developed an expression to determine the Poynting effect in isotropic incompressible materials, for which an explicit expression was given for a neo-Hookean material. Earlier works on second-order torsion such as those of Bhargava and Gupta (1976, 1979) and Blackburn and Green (1957) did not study the Poynting effect or the problem of combined torsion-axial loading. Green and Shield (1951) examined the problem of a small twist superposed on finite extension, and again the Poynting effect was not the objective of their investigation.

To our knowledge, the problem of the Poynting effect in multilayered composites has received little attention. On the other hand, many natural biomaterials such as skin, heart valves, articular cartilage, small-intestinal submucosa and vascular tissues are multilayered, see for example Diridollou et al. (2000), Sofer et al. (2002), Alexopoulos et al. (2003), Stella and Sacks (2007), Zhao et al. (2011) and Browning et al. (2012). Criscione et al. (1999) investigated the twist rate of cardiac capillary muscles under a constant axial stretch as a function of the twisting moment. The muscles are considered as composites consisting of a core of myocardium inside a thin sheath of endocardium. Another example is the stroma of the cornea, made up of collagen fibrils embedded in a hydrated matrix of proteoglycans, glycoproteins and keratocytes. The biomechanical response plays a critical part in corrective surgeries such as LASIK and diseases such as keratoconus and corneal dystrophies (Boyce et al., 2007). In general, connective tissues such as tendons and ligaments are fiber-reinforced soft composite materials, and the issues of nonlinearity, finite strains, anisotropy and viscoelasticity have been active areas of research (Limbert and Taylor, 2002). Furthermore, multilayered hydrogel networks can also be synthesized (Cuchiara et al., 2010) for many applications such as tissue regeneration, wound healing and drug delivery (Detzel et al., 2011; Jessel et al., 2006). Zhu et al. (2004) used a polyelectrolyte multilayer technique to create multilayer coating on poly-(DL-lactide) to promote chondrocyte attachment and growth. A layer-by-layer technique was also used to create multilayer coatings of chitosan and heparin on biomaterials to control MG-63 osteoblast adhesion and growth (Kirchhof et al., 2009). A multilayer biomimetic scaffold mimicking the structure of cancellous and cortical bones was designed for bone tissue engineering, which shows an enhanced mechanical strength and larger pore size in the center (Kong et al., 2007). In a physiological environment, biomaterials are often subjected to complex loading which may involve torsion and axial loading (tension or compression). The combined torsion-tension loading of a multilayered cylinder is thus of major interest.

In this paper, the analytical solutions for cylindrical multilayered composites under pure torsion, pure axial loading and combined torsion-axial loading are obtained within the framework of Murnaghan's (1951) second-order elasticity theory. The focus of the work is on the Poynting effect in soft and metallic composites. A further interesting question is, in contrast to the Poynting effect described thus far, whether an axial loading can contribute to the twist of the cylinder. We shall call this the "axial force–twist effect." The Poynting effect refers to the axial deformation of a cylinder under pure torsion, and was named in honor of Poynting (1909). This effect can be positive or negative. The axial force–twist effect can also be positive or negative, the former meaning that both the torsion and axial loadings produce twist in the same sense while the latter meaning that they produce twist in mutually opposite sense. We further note that under pure axial loading of a circular cylinder, the twist or rotational displacement is necessarily zero because of the axisymmetric constraint. This means that the Poisson effect for this displacement component does not exist in the usual sense. The Poisson effect exists for the radial displacement under pure axial loading. Hence, the axial force–twist effect, which is judged by the rotational displacement, is not associated with the Poisson effect. The axial force–twist effect is rarely discussed in the literature, an exception being Zubov (2001), who investigated it in terms of an isotropic incompressible neo-Hookean material. The term "inverse Poynting effect" was used by Zubov (2001), but it should not be confused with the term "negative Poynting effect".

The contributions of the current work can be summarized as follows: (1) consideration of bi-layered composites under torsion, axial and combined loadings, (2) elucidation of the second-order effects on mechanical behavior, (3) identification of the coupled components of the elastic fields under combined loading, (4) development of an explicit material- and geometry-dependent parameter for predicting the Poynting effect, (5) investigation of how material properties and geometrical configurations can influence the nonlinear phenomena, thus opening the possibility of designing bio-inspired multilayered composites with desirable characteristics, (6) exploration of the axial force–twist effect, and (7) determination of the third-order elastic constants of soft materials through the Poynting effect dependence on the radii of the constituent layers.

The paper is organized in following manner. In Section 2, the second-order nonlinear elastic model is established and analytical solutions are obtained for bi-layered cylinders under torsion, axial and combined loadings and specialized for (monolayered) homogeneous cylinders. In Section 3, the effects of material nonlinearity and geometrical complexities are presented, highlighting nonlinear phenomena such as the Poynting effect and the axial force–twist effect, as well as the existence of second-order normal stresses. Further discussions, especially for applications of results, are given in Section 4. A summary of the present work is given in Section 5.

2. Formulation and solutions

Consider an N-layered circular cylinder of length L and radius r_N (or simply R for a homogeneous cylinder). The interfaces are located at the radial coordinates $r = r_i$, $i = 1, N-1$, with $i = 1$ denoting the innermost interface, as shown in Fig. 1. All the layers are nonlinear elastic, isotropic, homogeneous and perfectly bonded to each other. The composite cylinder is subjected to either a torsion T, an axial loading P (tension or compression), or a combined T–P, with respect to the longitudinal direction. The initial coordinates of a particle of the cylinder are chosen as (r,θ,z), which respectively represent the radial, angular and axial coordinates. The final coordinates are denoted by (ρ,ψ,ζ).

Murnaghan (1951) has investigated the pure torsion, and partially the pure tensile loading, of a homogeneous cylinder. Besides the second-order elasticity framework of Murnaghan, one may adopt various forms of the strain energy density functions, e.g., neo-Hookean, Mooney-Rivlin, St. Venant–Kirchhoff, Ogden, and polynomial forms. The Murnaghan energy density function is chosen for the following reasons: (1) it is general rather than specific and the linear and second-order nonlinear effects can be distinguished; (2) the elastic properties are captured by two second-order and three third-order elastic constants, which can be determined through measuring speed variations of shear waves as a function of applied stress (Catheline et al., 2003); (3) analytical solutions can be obtained for multilayered geometries as found in this work; and (4) the elastic parameters can be correlated to