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Stability boundaries of two-parameter non-linear elastic structures

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ABSTRACT

The load-bearing capacity of structures can be influenced by variations in parameters, such as initial geometric defects, multi-parameter loadings, material specifications and temperature. This paper aims to introduce a new formulation to trace the stability boundaries of two-parameter elastic structures. The proposed procedure can find a set of critical points, both limit and bifurcation ones, via a modified Newton's method. In the authors' formulation, the residual force is set to zero, and a critically constraint is satisfied simultaneously. Numerical examples presented in this paper demonstrate the efficiency of the suggested method.

1. Introduction

In many structural systems, parameters, such as initial geometric defects, extra loadings and changes in temperature can significantly influence the load carrying capacity. The equilibrium path and, subsequently, the buckling strength are usually sensitive to these parameters (or imperfections). A broad class of structures, like columns, trusses, shallow arches and thin-walled structures are examples of such systems (Ikeda and Ohsaki, 2007; Parente et al., 2008). Finding a precise relationship between control parameters and the final strength of structures subjected to external loadings can be helpful for both analysts and structural designers to have better understanding about the structural behavior. In mechanical–structural problems, it is common to assume that the magnitude of imperfections varies through one or more control parameters (Huseyin, 1975). Subsequently, the equilibrium equations and the critical load(s) are dependent on these parameters.

Critical points (e.g. limit, simple bifurcation and multi-bifurcation points) play an important role in the post-buckling behavior of structures. Along tracing the equilibrium path, finding the type and the exact locus of such points is needed for choosing a suitable numerical strategy. In the literature, several techniques for the calculation of equilibrium paths are extensively discussed (Crisfield, 1983; Forde and Stiemer, 1987; Riks, 1979). Most of these numerical techniques are based on Newton's method, which gives a number of discrete equilibrium points through an incremental–iterative procedure (Chen and Blandford, 1993; Rezaiee-Pajand et al., 2009; Widjaja, 1998). Many of these techniques become divergent or choose a wrong path when they reach critical points. Previously, many efforts have been made by researchers in the area of critical points' detection (Battini et al., 2003; Seydel, 1979; Wriggers et al., 1988). Since the tangent stiffness matrix becomes singular at these points, most of the proposed methods use this characteristic as the critically constraint, which is added to the governing equations, and apply an iterative procedure to obtain the supposed critical point (Fujii and Ramm, 1997; Kouhia et al., 2012; Wriggers and Simo, 1990).

The final strength of a structure can be affected by control parameters (or imperfections), such as initial geometric defects, load imperfection and thermal stresses (Ohsaki and Ikeda, 2009; Parente et al., 2006). By simultaneously perturbing the equilibrium equations and the critically constraint in the vicinity of the critical point, the sensitivity analysis of critical states can be investigated (Godoy and Banchio, 2001; Thompson and Hunt, 1973; Wu and Wang, 1997). Although this type of method is compatible with the finite element coding, it needs the calculation of high-order derivatives of the tangent stiffness matrix to obtain a better result. Furthermore, the range of validity is restricted in the vicinity of the critical point. The Lyapunov–Schmidt–Koiter asymptotic approach is another technique with similar advantages and disadvantages. In this process, the governing equations are regularized by a perturbation parameter (Casciaro et al., 1998; Casciaro et al., 1992, 2009; Koiter, 1945). There are also a number of techniques based on incremental–iterative procedures that directly obtain the critical point(s) of parameterized (imperfect) structures (Eriksson et al., 1999; Moghaddasie and Stanciulescu, 2013a; Wu, 2000). In such methods, the equilibrium equations and the critically

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constraint are simultaneously convinced via an iterative procedure. The superiority of these schemes in comparison with perturbation approaches is that errors will not increase for large values of the parameter(s).

This paper introduces a new formulation to find the relationship between the buckling strength of non-linear elastic structures and the variation in a control parameter. In this way, an incremental–iterative procedure is used to simultaneously set the residual force to zero and convince the critical constraint. This constraint deals with the critical eigenvector of the tangent stiffness matrix. *In addition, the authors propose a formula to update the spherical arc-length constraint in each increment to improve the convergence.* The suggested technique is based on Newton's method, and leads to a set of discrete critical point. Each point is directly computed from the previous one. Consequently, the suggested approach is suitable for conservative systems (e.g. elastic structure), which the locus of critical points are independent of the relative equilibrium paths. The suggested method includes the following features all together: *(a) the mode change in buckling does not lead to divergence; (b) errors will not increase for large magnitudes of control parameters; (c) both limit and simple bifurcation points can be detected; (d) the convergence properties are not sensitive to variations in the stiffness matrix; and (e) since each critical point is directly calculated from the previous one, globalization techniques (which are necessary for the computation of the critical point from the unloaded state) are not needed to use for structures with large pre-critical displacements.* Applying a globalization technique is crucial when the current state is far from the desired critical point, and make the method convergent (see, for example, (Dennis Jr. and Schnabel, 1996)).

In the following, a brief outline for the paper is given: Section 2 provides some basic equations for tracing the equilibrium path. In addition, the spherical arc-length is briefly described. In Section 3, the characteristics of critical states are investigated, and a classification of simple critical points is introduced. Moreover, an iterative procedure for calculating the critical load from the unloaded state is presented. Section 4 defines the concept of stability boundary in parameterized systems. The formulation and the numerical implementation of the proposed method for parameter sensitivity analysis of critical points are given in this section. Numerical examples in Section 5 examine the accuracy and computational efficiency of the suggested procedure in tracing critical points with different types of control parameters and imperfections. Finally, concluding remarks are presented in Section 6.

2. Equilibrium path

The total potential energy Π is a function of the nodal displacement vector $\mathbf{u} \in \mathbb{R}^n$ and the load parameter $p \in \mathbb{R}$ for perfect structures. Here, n denotes the number of degrees of freedom (DoFs). This energy is a summation of internal strain energy Φ and the work done by the external load. For structures under a displacement independent loading, Π is:

$$\Pi(\mathbf{u}, p) = \Phi(\mathbf{u}) - p\,\mathbf{q}^T\mathbf{u}, \tag{1}$$

where $\mathbf{q}$ is the external load vector, and the superscript T shows the transpose of the supposed vector or matrix. In elastic structures, the value of Π is stationary for equilibrium states. Consequently, its first derivative with respect to $\mathbf{u}$, which is called residual force $\mathbf{r}$, is equal to $\mathbf{0}$, and it leads to a set of equilibrium equation as follows:

$$\mathbf{r}(\mathbf{u}, p) = \mathbf{F}_{int}(\mathbf{u}) - p\,\mathbf{q} = \mathbf{0}. \tag{2}$$

Here, $\mathbf{F}_{int}(\mathbf{u})$ represents the nodal internal force and equals $\partial\Phi/\partial\mathbf{u}$. The vector $p\mathbf{q}$ denotes the external load. The set of points satisfying Eq. (2) is called the *equilibrium path*.

In order to trace the equilibrium path, many numerical techniques have been developed and used in the literature (see, for example, (Crisfield, 1981; Krenk, 1995; Le Grognec and Le Van, 2008; Riks, 1979)). A robust scheme, which obtains a set of discrete points on the equilibrium path, is based on Newton's method. This method usually includes incremental and iterative parts. In this paper, $\Delta\mathbf{u}$ and Δp represent the nodal displacement and load increments (predictors), respectively, and their relationship is:

$$\mathbf{K}_T(\mathbf{u})\Delta\mathbf{u} = \Delta p\mathbf{q}, \tag{3}$$

where $\mathbf{K}_T$ denotes the tangent stiffness matrix and can be derived from the second derivative of the strain energy with respect to $\mathbf{u}$. In the iterative part, the increments are updated by correctors:

$$\begin{cases} \Delta\mathbf{u}_{i+1} = \Delta\mathbf{u}_i + \delta\mathbf{u}_i \\ \Delta p_{i+1} = \Delta p_i + \delta p_i. \end{cases} \tag{4}$$

The superscript i represents the iteration number within each increment. Since the incremental–iterative methods obtain a set of discrete points, an extra constraint is added to the system:

$$\begin{Bmatrix} \mathbf{r}(\mathbf{u}, p) \\ L(\mathbf{u}, p) \end{Bmatrix}_{(n+1)\times 1} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix}_{(n+1)\times 1}. \tag{5}$$

The analyst may utilize various formulae for L. In the arc-length algorithm, the additional constraint is assumed to be an $n+1$ dimensional sphere in the space of $(\mathbf{u}, p) \in \mathbb{R}^{n+1}$ (Crisfield, 1991):

$$\alpha_u^2\Delta\mathbf{u}^T\Delta\mathbf{u} + \alpha_p^2\Delta p^2 - \Delta s^2 = 0, \tag{6}$$

where Δs is the arc-length. The parameters α_u and α_p determine the contributions of displacement and load terms in the arc-length equation. Fig. 1 shows the incremental–iterative procedure in the arc-length approach. As it can be seen, $\Delta s/\alpha_u$ and $\Delta s/\alpha_p$ represent the radii of the $n+1$ dimensional sphere in the directions of $\mathbf{u}$ and p, respectively.

If the contribution of the load term in Eq. (6) is omitted by choosing $\alpha_p = 0$, the cylindrical arc-length constraint is obtained (Crisfield, 1981; Magnusson and Svensson, 1998; Ramm, 1981). This means that the radius of the $n+1$ dimensional sphere in the direction of the load parameter becomes infinitely large. In contrast, for the choice $\alpha_u = 0$, the spherical arc-length method changes into the standard Newton–Raphson (load control) scheme. Based on Fig. 1, the values of the first increments are relative to the magnitude of the arc-length Δs, and can be calculated as follows:

$$\Delta p_1 = \pm\frac{\Delta s}{\sqrt{\alpha_u^2\mathbf{b}_0^T\mathbf{b}_0 + \alpha_p^2}}, \tag{7}$$

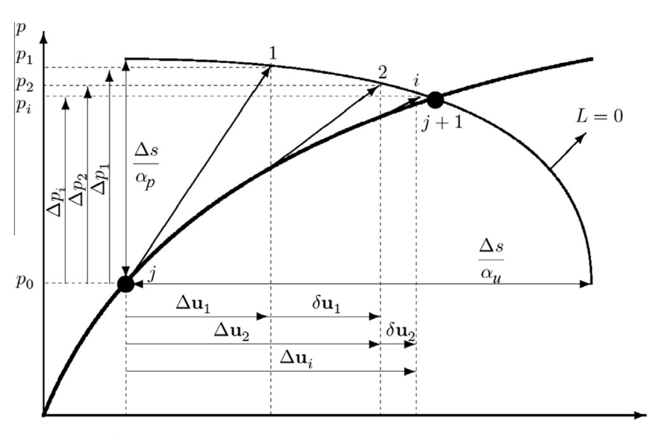

Fig. 1. Spherical arc-length procedure.