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Determination of stress intensity factors by the finite element discretized symplectic method

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ABSTRACT

When rewriting the governing equations in Hamiltonian form, analytical solutions in the form of symplectic series can be obtained by the method of separation of variable satisfying the crack face conditions. In theory, there exists sufficient number of coefficients of the symplectic series to satisfy any outer boundary conditions. In practice, the matrix relating the coefficients to the outer boundary conditions is ill-conditioned unless the boundary is very simple, e.g., circular. In this paper, a new two-level finite element method using the symplectic series as global functions while using the conventional finite element shape functions as local functions is developed. With the available classical finite elements and symplectic series, the main unknowns are no longer the nodal displacements but are the coefficients of the symplectic series. Since the first few coefficients are the stress intensity factors, post-processing is not required. A number of numerical examples as well as convergence studies are given.

1. Introduction

Fracture can occur in engineering components subject to mechanical loading conditions. stress intensity factors (SIF) are important parameters in design. It is well known that the finite element method (FEM) is the most general method for finding SIFs. However, FEM requires special techniques and a special mesh mapping in the vicinity of the crack tip for fracture mechanics applications and is generally extremely time consuming and is not high in precision in the vicinity at cracks. For these reasons, many variants of FEM for evaluating the SIFs have been developed.

Tong et al. (1973) first presented a hybrid singular element based on the Hellinger–Reissner variational principle in order to overcome the shortcomings of the conventional finite elements. After that, Ping et al. (2008) and Chen and Ping (2009) studied the singular stress fields around the vertex of an anisotropic multi-material wedge and the inplane singular elastic field problems of inclusion corners by a super singular wedge tip element. Yao and Hu (2011) presented a novel singular finite element to study cracked plates with arbitrary traction acting on crack surfaces. Karihaloo and Xiao (2001a,b) and Karihaloo et al. (2003) developed a higher-order hybrid crack element (HCE) to calculate the coefficients of higher order terms of the crack tip asymptotic field. The coefficients in standard fracture test specimens such as three-point bend beams and wedge-splitting specimens were determined with the aid of HCE. Lin and Abel (1988) introduced a virtual crack extension technique that employs both a variational formulation and FEM to calculate the mode-I SIF for a structure containing a single crack. Subsequently, Hwang et al. (1998,2001,2005) and Hwang and Ingraffea (2007) generalized this method to study multiple crack systems and 3D planar cracks. The extended finite element method (XFEM) was originally proposed by Belytschko and Black (1999). They presented a method for enriching the finite element approximations so that crack growth problems can be solved with minimal re-meshing. Yazid et al. (2009) presented a review of XFEM for computational fracture mechanics and discussed the basic ideas and formulation for the newly developed XFEM method. Besides the above works, the fractal geometry concepts were introduced into finite element method. Reddy and Rao (2008a,b) used a fractal finite element method (FFEM) to analyze cracks in a homogeneous, isotropic, and two-dimensional linear-elastic body subject to mixed-mode (modes I and II) loading conditions. Su et al. (2003) and Su and Fok (2007) determined the coefficients of the crack tip asymptotic field by FFEM. Leung and Tsang (2000) and Tsang et al. (2003,2004) developed a FFEM for the analysis of static and dynamic crack problems. It can determine SIFs directly and proved to be very efficient and accurate.

In this paper, a finite element discretized symplectic method (FEDSM) is developed for calculating the stress-intensity factors in linear-elastic crack problems. The method separates the overall cracked elastic body into a finite size singular stress region near the

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crack tip and a regular region far away from the crack tip, i.e., near field and far field. Both the near and far fields were meshed by the conventional elements. A symplectic dual approach for elasticity which was first introduced by Zhong and co-workers (Yao et al., 2009; Lim and Xu, 2010; Zhong et al., 2009) is performed in the near fields. The symplecic method has been widely used in many research areas, e.g. theory of plates and shells (Li et al., 2011, 2013), fracture mechanics (Leung et al., 2009; Xu et al., 2010), viscoelasticity (Zhang and Xu, 2006), fluid mechanics (Wang et al., 2009), functional graded effects (Chen and Zhao, 2009), piezoelectricity (Leung et al., 2007), etc. The analytical solution around the crack tip in the near field is solved and expanded in terms of the symplectic eigenfunctions. The displacement, stress and SIFs can be analytically represented. Marking use of the analytical solution, a displacement transformation is introduced to the near field so that the large number of nodal displacements there can be reduced effectively to a small set of undetermined coefficients of the symplectic eigenfunctions. Consequently, computer storage and solution times are reduced significantly and parallel computation is possible. The remaining of the paper is organized in the following manner. In Section 2, the fundamental equations in Lagrangian form are transformed to the Hamiltonian form. In Section 3, the governing equations are solved by the method of separation of variables and the displacements and stresses are expressed analytically by the symplectic eigenfunctions. In Sections 4 and 5, the general formulation of FEDSM and SIFs for the crack systems are presented. Section 6 shows the numerical results and discusses the accuracy and efficiency.

2. The fundamental problem and Hamilton system

Consider an isotropic edge-cracked media in polar coordinates (r, θ). The r-axis is along the radial direction with the origin located at the tip of the crack as shown in Fig. 1. The overall cracked media is divided into near field and far field regions. The crack face is at $\theta = \pm\pi$ and the length of the crack is a. The constitutive relation and strain–displacement relation in the near fields are governed by

$$\sigma_r = E(\varepsilon_r + \upsilon\varepsilon_\theta)/(1-\upsilon^2), \quad \sigma_\theta = E(\varepsilon_\theta + \upsilon\varepsilon_r)/(1-\upsilon^2), \\ \tau_{r\theta} = E\varepsilon_{r\theta}/[2(1+\upsilon)] \tag{1}$$

$$\varepsilon_r = \partial_r u_r, \quad \varepsilon_\theta = u_r/r + 1/r\partial_\theta u_\theta, \quad \varepsilon_{r\theta} = \partial_r u_\theta - u_\theta/r + 1/r\partial_\theta u_r \tag{2}$$

where σ_{ij} and ε_{ij} be the components of stresses and strains, u_r and u_θ are displacements along the r-axis and θ-axis, E is the elastic modulus, υ is Poisson's ratio and $G = E/[2(1+\upsilon)]$ is the shear modulus.

In the following part, we rewrite the governing equation in the second order Lagrangian form to the first order Hamiltonian form

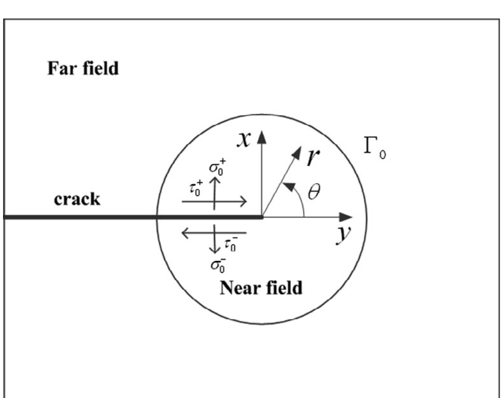

Fig. 1. An isotropic media with a single edge crack.

so that the method of separation of variables can be formally applied to find both the stress and displacement distributions analytically in symplectic eigenfunctions. Introduce the transformation $\eta = \ln r$ and use over-dot to represent differentiation with respect to η, namely $\dot{()} = \partial()/\partial\eta$. Define the original variable $\mathbf{q}$ and dual variable $\mathbf{p}$ as

$$\mathbf{q} = \{ u_r \quad u_\theta \}^{\mathrm{T}} \quad \text{and} \quad \mathbf{p} = \{ r\sigma_r \quad r\tau_{r\theta} \}^{\mathrm{T}}, \tag{3}$$

respectively corresponding to the configuration and momentum variables in the classical Hamiltonian analysis. The Hamiltonian equations is given by

$$\dot{\mathbf{\Psi}} = \mathbf{H\Psi} + \mathbf{f} \tag{4}$$

where $\mathbf{\Psi} = \{ \mathbf{q} \quad \mathbf{p} \}^{\mathrm{T}}$, $\mathbf{H}$ and $\mathbf{f}$ are the Hamiltonian operator matrix and the non-homogenous part given by

$$\mathbf{H} = \begin{bmatrix} -\upsilon & -\upsilon\partial_\theta & (1-\upsilon^2)/E & 0 \\ -\partial_\theta & 1 & 0 & 2(1+\upsilon)/E \\ E & E\partial_\theta & \upsilon & -\partial_\theta \\ -E\partial_\theta & -E\partial_\theta^2 & -\upsilon\partial_\theta & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = -e^{2\eta} \begin{Bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \end{Bmatrix}. \tag{5}$$

The eigenfunctions of the Hamiltonian operator matrix have some distinguished behaviors as mentioned in Leung et al. (2009). The eigensolutions of the Hamiltonian operator matrix have some particular behaviors that if μ_j is an eigenvalue, $-\mu_j$ is an eigenvalue also. Hence the eigensolutions can be subdivided into two groups of α with positive real part and β with negative real part so that

$$(\alpha): \quad \mu_j^{(\alpha)}, i = 1, 2, \ldots, \quad \mathrm{Re}(\mu_j^{(\alpha)}) > 0 \text{ or } \mathrm{Re}(\mu_j^{(\alpha)}) = 0 \text{ and } \mathrm{Im}(\mu_j^{(\alpha)}) > 0$$

$$(\beta): \quad \mu_j^{(\beta)}, i = 1, 2, \ldots, \quad \mu_j^{(\beta)} = -\mu_j^{(\alpha)}$$

whose eigenfunction-vectors are denoted respectively as $\psi_j^{(\alpha)}$ and $\psi_j^{(\beta)}$. Introducing an inner product $\langle \psi_i, \mathbf{J}, \psi_j \rangle = \int_\Omega (\mathbf{q}_i\mathbf{p}_j - \mathbf{q}_j\mathbf{p}_i) d\theta$ between any two of them, one has the adjoint symplectic orthonormal relations

$$\langle \psi_n^{(\alpha)}, \mathbf{J}, \psi_k^{(\alpha)} \rangle = \langle \psi_n^{(\beta)}, \mathbf{J}, \psi_k^{(\beta)} \rangle = 0 \\ \langle \psi_n^{(\alpha)}, \mathbf{J}, \psi_k^{(\beta)} \rangle = \delta_{nk}, \quad \langle \psi_n^{(\beta)}, \mathbf{J}, \psi_k^{(\alpha)} \rangle = -\delta_{nk}$$

where δ_{ij} is the Kronecker delta which equals to one if $i = j$ and equals to zero otherwise. $\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$ for the identity matrix $\mathbf{I}$.

The boundary conditions along the crack surface are:

$$\begin{cases} [E(u_r + \partial_\theta u_\theta)/r + \upsilon\sigma_r/r]_{\theta=\pm\pi} = \sigma_\theta^\pm \\ \tau_{r\theta}|_{\theta=\pm\pi} = \tau_{r\theta}^\pm \end{cases} \tag{6}$$

Here, $\sigma_\theta^\pm$ and $\tau_{r\theta}^\pm$ are the surface tractions along the crack surfaces $\theta = \pm\pi$.

3. The symplectic eigenvalue and eigenfunctions

Consider the homogeneous part of Eq. (4) with the traction free crack conditions at the inner boundary surfaces. The eigenvalue eigenfunctions can be obtained similar to Leung et al. (2009) and are divided into two groups: zero eigenfunctions (the eigenfunctions having zero eigenvalue) and non-zero eigenfunctions (otherwise).

3.1. The zero eigenfunctions

Because of the traction-free natural boundary conditions, there exist zero eigenvalues whose eigenfunctions correspond to the