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Finite elastic deformations of transversely isotropic circular cylindrical tubes

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ABSTRACT

We consider the finite radially symmetric deformation of a circular cylindrical tube of a homogeneous transversely isotropic elastic material subject to axial stretch, radial deformation and torsion, supported by axial load, internal pressure and end moment. Two different directions of transverse isotropy are considered: the radial direction and an arbitrary direction in planes normal locally to the radial direction, the only directions for which the considered deformation is admissible in general. In the absence of body forces, formulas are obtained for the internal pressure, and the resultant axial load and torsional moment on the ends of the tube in respect of a general strain-energy function. For a specific material model of transversely isotropic elasticity, and material and geometrical parameters, numerical results are used to illustrate the dependence of the pressure, (reduced) axial load and moment on the radial stretch and a measure of the torsional deformation for a fixed value of the axial stretch.

1. Introduction

It is well known from the literature (see, for example, the review by Saccomandi (2001)) that, in the absence of body forces, a number of deformations can be supported in equilibrium in an incompressible isotropic nonlinearly elastic solid material by application of surface tractions alone. Such deformations are said to be *controllable*. If, within a given class of materials, the deformation is controllable for all materials and independent of any specific constitutive law in the considered class then the deformation is said to be *universal* (within the considered class). Saccomandi (2001) introduced the term *relative-universal* for situations where the class of materials is a subclass of a general class of materials. The deformation that is of particular interest in the present paper is a combined deformation consisting of the (i) finite extension, (ii) inflation and (iii) torsion of a cylindrical circular tube, which, for an isotropic material, is indeed universal. However, for anisotropic materials, in particular for the transversely isotropic materials with which we are concerned is this paper, this deformation is only controllable for certain directions of transverse isotropy, and then, in these cases, it is also universal.

Several authors have studied the deformations (i)–(iii) for isotropic materials from many different perspectives in the past. In brief, torsional deformations for incompressible isotropic materials were first examined in a series of papers by Rivlin (1948, 1949a,b) while associated experimental data were provided in Rivlin (1947) and Rivlin and Saunders (1951). Gent and Rivlin (1952), guided by the theoretical results of Rivlin (1949b), performed experiments to obtain data for the problem of combined uniform extension, uniform inflation and small amplitude torsion. Comparison of the Ogden model (Ogden, 1972) for rubberlike solids with the data given in Rivlin and Saunders (1951) for solid and tubular cylinders composed of natural rubber under combined extension and torsional deformation has been presented by Ogden and Chadwick (1972). A detailed analysis of the combined extension and inflation of such materials with particular reference to bifurcation into non-circular cylindrical modes of deformation was provided by Haughton and Ogden (1979a,b).

More recently, Horgan and Saccomandi (1999) used a material model incorporating limiting chain extensibility to capture the hardening response of incompressible isotropic elastic materials under large strain torsional deformations, while Kanner and Horgan (2008) were concerned with investigating the effects of strain stiffening on the response of solid circular cylinders in the combined deformation of torsion superimposed on axial extension.

For compressible isotropic materials, for which the deformations (i)–(iii) are not, in general, controllable, a class of materials admitting isochoric pure torsional deformation was proposed by Polignone and Horgan (1991). In the same spirit, Kirkinis and Ogden (2002) derived analogous solutions and also introduced a methodology for generating corresponding results for

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incompressible materials. Different aspects of pure torsion for special classes of compressible materials and considerations of loss of ellipticity have also been studied in Beatty (1996) and Horgan and Polignone (1995), respectively, amongst others.

The contributions mentioned above related to rubberlike materials, but more recently attention has also been focused on elastic deformations of soft biological tissues in the context of biomechanics, and these materials are in general anisotropic, typically transversely isotropic or orthotropic. To the best of our knowledge, very few authors have studied the deformations (i)–(iii) for *anisotropic* elastic solids in the finite deformation regime, in particular for incompressible transversely isotropic elastic solids, including fibre-reinforced materials, although Green and Adkins (1970) presented some general theoretical results for a transversely isotropic circular cylindrical tube subject to axial extension, inflation and torsion for the case in which the axis of transverse isotropy is aligned with the tube axis. Also, under the restriction of idealized fibre reinforcement (i.e. inextensible fibres), Spencer (1972) discussed the problem of extension and torsion of solid elastic cylinders augmented with one or two families of helical fibres, although the analysis is mainly restricted to the linear theory (see also the interesting discussion relating to two symmetric helically disposed fibre families in Spencer (1984)). For large deformations, in the context of soft tissue biomechanics (with particular reference to arteries), the problem of extension and inflation has been examined by Ogden and Schulze-Bauer (2000), with the anisotropy associated with helical fibre reinforcement, which is used to model the contribution of embedded collagen fibres to the overall response of the tissue, while Horgan and Saccomandi (2003) discussed the combined extension and inflation problem for soft tissues by taking into account limiting chain extensibility. A thorough analysis of the elastic response of arteries, for simultaneous extension, inflation and torsion, was provided by Holzapfel et al. (2000).

In the present analysis, we consider the problem of combined finite extension/contraction, radial contraction/expansion and torsion of a circular cylindrical tube of homogeneous elastic material with specific directions of transverse isotropy (which may be, but need not necessarily be considered as a material reinforced by a single family of fibres). In particular, in Section 2 we introduce the notation and summarize the necessary kinematics for the combined deformation in an incompressible material. We then summarize, in Section 3, the constitutive equation for a transversely isotropic material, and the equilibrium equations (in the absence of body forces) are used to obtain general formulas for the internal pressure in the tube, the resultant axial load and moment on the ends of the tube that are applied to maintain the prescribed deformation in respect of a general transversely isotropic form of constitutive law. These results, which also apply in the isotropic specialization, recover the formulas given in Haughton and Ogden (1979a), for the case in which no torsion is applied to the tube.

In Section 4 we highlight the fact that, for transversely isotropic materials, the considered deformation cannot be maintained for all possible directions of transverse isotropy, and we therefore specialize to those directions which are admissible, specifically the radial direction and directions locally lying in planes normal to the radius of the tube.

In general, closed form solutions are not obtainable in simple form, and in order to illustrate the results we therefore provide numerical results based on a simple prototype form of transversely isotropic strain-energy function in Section 5. In particular, we show, in graphical form, how, for a fixed value of the axial extension, the pressure, the (reduced) axial load and the moment depend on the applied torsion and radial stretch for a specific tube thickness and transverse isotropy parameter. Finally, a brief summary of the results is given in the concluding Section 6.

2. Kinematics and geometry

Consider a material continuum which, when unstressed and unstrained, occupies the *reference* configuration $\mathcal{B}_r$. Let a typical material point in this configuration be identified by its position vector $\mathbf{X}$. The corresponding position vector in the deformed configuration $\mathcal{B}$ is denoted $\mathbf{x}$ and the deformation from $\mathcal{B}_r$ to $\mathcal{B}$ is written $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where the vector function $\boldsymbol{\chi}$ is referred to as the *deformation* (we are considering quasi-static deformations here). The deformation gradient tensor, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}(\mathbf{X}), \tag{1}$$

where Grad is the gradient operator with respect to $\mathbf{X}$. The associated right and left Cauchy–Green deformation tensors, denoted $\mathbf{C}$ and $\mathbf{B}$ respectively, are defined as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \tag{2}$$

where $\mathbf{U}$ and $\mathbf{V}$, respectively, are the right and the left stretch tensors, which are positive definite and symmetric and come from the polar decompositions $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, $\mathbf{R}$ being a proper orthogonal tensor. For a homogeneous incompressible nonlinearly isotropic elastic solid, the elastic stored energy (defined per unit volume) depends on only two invariants, which are the principal invariants of $\mathbf{C}$ (equivalently of $\mathbf{B}$), defined by

$$I_1 = \mathrm{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \mathrm{tr}(\mathbf{C}^{-1}) = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{3}$$

where $\lambda_i > 0, i \in \{1, 2, 3\}$, are the principal stretches, i.e. the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$. The incompressibility constraint, which in terms of $\mathbf{F}$ is

$$\det\mathbf{F} = 1, \tag{4}$$

may be written in terms of the principal stretches as

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{5}$$

If the material has a single distinguished direction (the direction of transverse isotropy), identified by the unit vector $\mathbf{M}$ in the reference configuration, two more invariants, denoted I_4 and I_5 (in general independent), are introduced that are associated with $\mathbf{M}$. These invariants are defined by

$$I_4 = \mathbf{FM}\cdot\mathbf{FM} = \mathbf{m}\cdot\mathbf{m}, \quad I_5 = \mathbf{CM}\cdot\mathbf{CM} = \mathbf{m}\cdot\mathbf{Bm}, \tag{6}$$

where we have introduced the vector $\mathbf{m} = \mathbf{FM}$, which represents the direction of transverse isotropy in the deformed configuration. In general $\mathbf{m}$ is not a unit vector.

2.1. Combined extension, inflation and torsion

We now consider a circular cylindrical tube, which, in terms of cylindrical polar coordinate (R, Θ, Z), is defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L \tag{7}$$

in the reference configuration $\mathcal{B}_r$, where A and B are the internal and external radii and L is the length of the tube. The position vector $\mathbf{X}$ of a point of the tube is given by

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z, \tag{8}$$

where $\mathbf{E}_R$ and $\mathbf{E}_Z$ are the unit basis vectors associated with R and Z, respectively. We also denote by $\mathbf{E}_\Theta$ the corresponding unit vector associated with Θ.

The position vector $\mathbf{x}$ in the deformed tube is written

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z, \tag{9}$$

where we make use of cylindrical polar coordinates (r, θ, z) in $\mathcal{B}$, which are associated with unit basis vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$. The