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Direction dependent orthotropic model for Mullins materials

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ABSTRACT

The motivating key for this work was the absence of a phenomenological model that can reasonably predict a variety of non-proportional experimental data on the anisotropic Mullins effect for different types of rubber-like materials. Hence, in this paper, we propose a purely phenomenological direction dependent orthotropic model that can describe the anisotropic Mullins behaviour with permanent set and, has orthotropic invariants that have a clear physical interpretation. The formulation is based on an orthotropic principal axis theory recently developed for nonlinear elastic problems. A damage function and a direction dependent damage parameter are introduced in the formulation to facilitate the analysis of anisotropic stress softening in rubber-like materials. A direction dependent free energy function, written explicitly in terms of principal stretches, is postulated. The proposed theory is able to predict and compares well with experimental data available in the literature for different types of rubberlike materials.

1. Introduction

When subjected to cyclic loadings many rubberlike and biological materials exhibit an anisotropic stress-softening phenomenon widely known as the Mullins effect (Mullins, 1947). Due to its theoretical and technological interest, there is a wide literature on the Mullins effect; readers are referred to the literature (Shariff, 2000, 2006; Dargazany and Itskov, 2009; Merckel et al., 2012; Dorfmann and Pancheri, 2012) for detail description on the anisotropic behaviour of the Mullins effect.

Softening induced anisotropy is demonstrated by performing successive non-proportional loadings (i.e. successive loadings with changing the directions of stretching or the type of loading) and, recently, several non-proportional experiments (Hanson et al., 2005; Diani et al., 2006; Diani et al., 2006; Itskov et al., 2006; Dargazany and Itskov, 2009; Machado et al., 2012; Merckel et al., 2012) were conducted. However, only a few phenomenological models (Shariff, 2000, 2006; Itskov et al., 2006; Dorfmann and Pancheri, 2012; Merckel et al., 2013) describing anisotropic Mullins behaviour appeared in the literature. Except for Shariff (2006) model, the performances of previous phenomenological models were not tested against a wide range of deformations and different types of materials. We note that, Shariff (2006) direction dependent model describes the anisotropic behaviour in Mullins materials via a symmetric direction dependent orthotropic structural tensor $\boldsymbol{D}$ and a symmetric direction dependent structural shear tensor $\boldsymbol{S}$. His 2006 model does not consider permanent set and most of his anisotropic results were obtained using only the orthotropic tensor $\boldsymbol{D}$. His results compare well with the few anisotropic experimental data available at that time and were able to describe non-proportional loadings. However, the efficacy of his model cannot be further justified since there were very few non-proportional loading experiments existed before 2006. But since 2006, several non-proportional loading experiments appeared in the literature and, in view of this, to further justify the efficacy of Shariff (2006) model, a direction dependent orthotropic model is proposed in this paper. Although the proposed thermodynamically consistent phenomenological model is based on Shariff (2006) model, it is formulated differently. The formulation here does not use the structural tensors $\boldsymbol{D}$ and $\boldsymbol{S}$ but used a principal axis formulation, recently developed for orthotropic nonlinear elasticity (Shariff, 2011). It also takes permanent set into account. In this communication, the efficacy of the proposed model is tested against a wide range of non-proportional loadings and different types of rubber-like materials.

The paper is organised as follows. Since most readers are not familiar with the principal axis formulation recently developed by Shariff (2011), it is briefly outlined in Section 2. The direction dependent damage parameter and the damage function are proposed in Section 3. The materials in Sections 2 and 3 are used in Section 4 to formulate the constitutive equation and, in Section 5 energy dissipation is shown via the Clausius–Duhem inequality. In Section 6, the performance of the constitutive equation is tested

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against several types of loadings, including non-proportional loadings. Also in Section 6, the proposed theory is compared to several different types of experiments and rubber-like materials. The model in Section 4 is extended to take permanent set into account and, in Section 7, a permanent set model is proposed and its results are compared with experimental findings. For benefit of the readers the key equations are given in Appendix A. Finally, concluding remarks close the paper.

2. Principal axis formulation: nonlinear orthotropic elasticity

Since our model is based on a principal axis formulation developed for nonlinear orthotropic elasticity, in this Section, we briefly outline the preliminaries of the principal axis formulation recently developed by Shariff (2011). The principal stretch λ_i $(i = 1, 2, 3)$ is given by

$$\lambda_i = \boldsymbol{e}_i \bullet \boldsymbol{U}\boldsymbol{e}_i, \tag{1}$$

where $\boldsymbol{U}$ is the right stretch tensor and $\boldsymbol{e}_i$ is a principal direction of $\boldsymbol{U}$. In this communication, all subscripts i and j take the values 1, 2 and 3, unless stated otherwise. In Shariff (2011) paper, a strain energy function W_e for an incompressible orthotropic material is proposed, where its invariants have immediate physical interpretation. It has the form

$$\begin{aligned} W_e &= W(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) \\ &= \tilde{W}\left(\lambda_1, \lambda_2, \lambda_3 = \frac{1}{\lambda_1 \lambda_2}, \zeta_1, \zeta_2, \xi_1, \xi_2\right), \end{aligned} \tag{2}$$

where the invariants $1 \geqslant \zeta_i = (\boldsymbol{a} \bullet \boldsymbol{e}_i)^2 \geqslant 0$ and $1 \geqslant \xi_i = (\boldsymbol{b} \bullet \boldsymbol{e}_i)^2 \geqslant 0$ and, the perpendicular vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are the preferred orthotropic directions. The physical meaning of λ_i is obvious and it is clear that ζ_i and ξ_i are the square of the cosine of the angle between the principal direction $\boldsymbol{e}_i$ and the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

The function W enjoys the symmetry

$$W(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) = W(\lambda_2, \lambda_1, \zeta_2, \zeta_1, \xi_2, \xi_1). \tag{3}$$

A specific form of (3) has been proposed to characterise the mechanical behaviour of passive myocardium (Shariff, 2013).

The classical invariants I_k, $(k = 1, 2, \ldots, 7)$ are related to the physical invariants via the relations

$$\begin{aligned} I_1 &= tr\boldsymbol{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{(tr\boldsymbol{C})^2 - tr\boldsymbol{C}^2}{2} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \\ I_4 &= \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a} = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3, \\ I_5 &= \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a} = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3, \\ I_6 &= \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b} = \lambda_1^2\xi_1 + \lambda_2^2\xi_2 + \lambda_3^2\xi_3, \\ I_7 &= \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b} = \lambda_1^4\xi_1 + \lambda_2^4\xi_2 + \lambda_3^4\xi_3, \end{aligned} \tag{4}$$

where the right Cauchy–Green deformation tensor $\boldsymbol{C} = \boldsymbol{U}^2$, $\zeta_3 = 1 - \zeta_1 - \zeta_2$ and $\xi_3 = 1 - \xi_1 - \xi_2$. For an incompressible solid, the invariant $I_3 = det(\boldsymbol{C}) = (\lambda_1\lambda_2\lambda_3)^2 = 1$. It is shown by Shariff (2013) that the invariant sets $\{I_1, I_2, I_4, I_5, I_6, I_7\}$ and $\{\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2\}$ are a minimal integrity basis (Spencer, 1971) with a syzygy; only five of these invariants are independent. The second Piola–Kirchhoff stress is given by

$$\boldsymbol{T}^{(2)} = 2\frac{\partial W_e}{\partial \boldsymbol{C}} - p\boldsymbol{C}^{-1}, \tag{5}$$

where p is the Lagrange multiplier associated with the incompressible constraint $\lambda_1\lambda_2\lambda_3 = 1$. Principal axis formulation requires the symmetric components $(\frac{\partial W_e}{\partial \boldsymbol{C}})_{ij}$ of $\frac{\partial W_e}{\partial \boldsymbol{C}}$ relative to the basis $\{\boldsymbol{e}_i\}$. These components are (Shariff, 2011)

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial \tilde{W}}{\partial \lambda_i} \quad (i \text{ not summed}) \tag{6}$$

and the shear components

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij} = \frac{1}{\lambda_i^2 - \lambda_j^2}\left(\left(\frac{\partial \tilde{W}}{\partial \zeta_i} - \frac{\partial \tilde{W}}{\partial \zeta_j}\right)\boldsymbol{e}_i \bullet \boldsymbol{A}\boldsymbol{e}_j + \left(\frac{\partial \tilde{W}}{\partial \xi_i} - \frac{\partial \tilde{W}}{\partial \xi_j}\right)\boldsymbol{e}_i \bullet \boldsymbol{B}\boldsymbol{e}_j\right),$$

$$i \neq j, \quad i,j = 1, 2, \tag{7}$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{\alpha 3} = \frac{1}{\lambda_\alpha^2 - \lambda_3^2}\left(\frac{\partial \tilde{W}}{\partial \zeta_\alpha}\boldsymbol{e}_\alpha \bullet \boldsymbol{A}\boldsymbol{e}_3 + \frac{\partial \tilde{W}}{\partial \xi_\alpha}\boldsymbol{e}_\alpha \bullet \boldsymbol{B}\boldsymbol{e}_3\right), \quad \alpha = 1, 2, \tag{8}$$

where $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$ and $\boldsymbol{B} = \boldsymbol{b} \otimes \boldsymbol{b}$ ($\otimes$ denotes the dyadic product). It is assumed that $\tilde{W}$ has sufficient regularity to ensure that, as λ_i and λ_α approach λ_j and λ_3, respectively, Eqs. (7) and (8) have limits. It is explicit in Eqs. (7) and (8) that the second Piola–Kirchhoff stress is coaxial with $\boldsymbol{C}$ when the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel to any two of the principal directions. This explicitness may not be as transparent if the strain energy function is expressed in terms of the classical invariants (4) (or possibly most types of invariants found in the literature). The Cauchy stress us given by

$$\boldsymbol{\sigma} = 2\boldsymbol{F}\frac{\partial W_e}{\partial \boldsymbol{C}}\boldsymbol{F}^T - p\boldsymbol{I}. \tag{9}$$

3. Direction dependent damage parameter and damage function

In this paper the term "damage" is interpreted in its widest sense; for example, it may mean "rupture of molecular bonds that reform to create new microstructure" or "conversion of hard phase to soft phase" or "cavitation damage" or "any change in the ground state mechanical properties that are induced by strain". We are only concerned with strain induced damages that lead to stress softening. A damage function is introduced to measure an amount of damage caused by strain. A measure of damage is an important tool for analysing stress-softening materials (Shariff, 2006, 2009; Ogden and Roxburgh, 1999) The proposed damage function g (which may depend on material properties) is defined such that $0 = g(\boldsymbol{1}) \leqslant g(\boldsymbol{u}), \boldsymbol{u} \in T = \{\boldsymbol{u} = [u_1, u_2, \ldots u_n]^T \in R^n,\ u_k > 0, k = 1, 2, \ldots n\}$. The function g has also the properties that $\hat{g}'(\alpha) \geqslant 0$, where $\hat{g}(\alpha) = g((1-\alpha)\boldsymbol{1} + \alpha\boldsymbol{w}), 0 < \alpha \leqslant 1$ and $\boldsymbol{w}(\neq \boldsymbol{1}) \in T$ is a constant. $\hat{g}'(\alpha)$ may or may not exist at $\alpha = 0$. If it exist then $\hat{g}'(0) = 0$. In view of our definition, g increases monotonically as $\boldsymbol{u}$ moves away in an n-dimensional straight line from the point $\boldsymbol{u} = \boldsymbol{1}$. It is possible that, in order to adequately describe stress-softening behaviour in a compressible solid, a constitutive equation may consist of more than one forms of damage function. In this paper, we are only concerned with one-dimensional $\boldsymbol{u}$, i.e., $\boldsymbol{u} = x \in R$ and $x > 0$. If, for example, for a particular material, the compressing of an $\boldsymbol{e}_i$-line element does not contribute to stress softening, then we can construct g such that $g(x) = 0$ for $x < 1$. In this communication, however, we propose g to have the form

$$g(x) = \frac{(x^{c_1} - 1)^2}{x^{c_2}}, \tag{10}$$

where c_1 and c_2 are material constants and they must be constrained so that $g(x)$ increases monotonically as x moves away from the point $x = 1$; we note that the inequality $2c_1 > c_2$ ensures the monotonicity of g.

In order to relate a direction dependent *damage parameter*, on a line element parallel to a principal direction $\boldsymbol{e}_i$, to a mechanical value, we consider the following inequality

$$s_i^{(min)} \leqslant \lambda_i \leqslant s_i^{(max)}, \tag{11}$$