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The effects of particle dynamics on the calculation of bulk stress in granular media

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ABSTRACT

Expressions for bulk stress within a granular material in a dynamic setting are reviewed and explicitly derived for assemblies of three dimensional arbitrary shaped particles. By employing classical continuum and rigid body mechanics, the mean stress tensor for a single particle is separated into three distinct components; the familiar Love–Webber formula describing the direct effect of contacts, a component due to the net unbalanced moment arising from contact and a symmetric term due to the centripetal acceleration of material within the particle. A case is made that the latter term be ignored without exception when determining bulk stress within an assembly of particles. In the absence of this centripetal term an important observation is made regarding the nature of the symmetry in the stress tensor for certain types of particles; in the case of particles with cubic symmetry, the effects of dynamics on the bulk stress in an assembly is captured by an entirely skew-symmetric tensor. In this situation, it is recognised that the symmetric part of the Love–Webber formula is all that is required for defining the mean stress tensor within an assembly – regardless of the dynamics of the system.

1. Introduction and background

In recent times the homogenisation processes involved in developing a continuous, macroscopic definition of stress in a discrete granular material have received a great deal of attention. At one level the definition of stress is quite straight-forward and generally uncontested, however the area has spawned its fair share of historical debate. Most of this debate surrounded the potential for asymmetry in the stress tensor (Bardet and Vardoulakis, 2001; Bagi, 2003; Kuhn, 2003; Bardet and Vardoulakis, 2003a,b; Froiio et al., 2006) and arguments for the existence of 'couple stresses' in the granular continuum (Chang et al., 1990; Oda, 1999; Oda et al., 2000; Ehlers et al., 2003; Ehlers, 2010; Alonso-Marroquin, 2011; Goldhirsch, 2010). This is now largely resolved with general acceptance that, without contact moments, asymmetry does not exist in equilibrium (de Saxcé et al., 2004; Fortin et al., 2003). Some recent work has even demonstrated that asymmetry is not necessarily inherent in the presence of contact moments, provided such moments are properly accounted for in the homogenisation process (Wensrich, 2014).

The majority of prior work has focused on material in equilibrium or in a quasi-static state. Here a variety of approaches have been used such as the mean stress theorem as initially applied by Love (1927) and Weber (1966), course graining (e.g. Goldhirsch, 2010; Edwards and Grinev, 1999; Weinhart et al., 2012) and variational methods (i.e. virtual work, Bardet and Vardoulakis, 2001; Chang et al., 2005; Mehrabadi et al., 1982; Christoffersen et al., 1981; Satake, 1983; Goddard, 2007). From the perspective of these static approaches, apparent asymmetry can arise if the assumption of equilibrium is violated. In response there has been a significant amount of recent work focused on developing consistent homogenisation processes that are applicable even in the absence of equilibrium. These approaches largely focus on calculating stress as an ensemble average of the stress within individual particles, defined from the point of view of the conservation of momentum at all points within a given particle (de Saxcé et al., 2004; Fortin et al., 2003; Fortin et al., 2002; Li et al., 2009; Nicot et al., 2013; Moreau, 2010; Luding, 2010). This work has shown that the components of stress arising from particle dynamics may be significant and are necessary for eliminating the asymmetry present in earlier quasi-static descriptions.

In this paper, we apply a similar approach to define stress within a dynamic granular assembly as an ensemble average over individual particles subject to the laws of classical continuum theory and rigid body mechanics. For the most part, the approach taken here is known (de Saxcé et al., 2004; Fortin et al., 2003; Fortin et al., 2002; Li et al., 2009; Nicot et al., 2013; Moreau,

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2010; Luding, 2010), however we consider three-dimensions and arbitrary particle shape and progress to the point where a new interpretation can be made. In particular, we present an argument that the component of stress relating to the angular velocity of particles, often referred to as "centrifugal stress" (de Saxcé et al., 2004; Nicot et al., 2013), be excluded without exception from the definition of bulk stress. Without this term, we make further observations on the nature of stress symmetry in granular materials that will greatly simplify the process of accounting for dynamics in certain classes of materials, including the vast majority of Discrete Element Models where spherical particles are still commonplace.

2. Bulk stress in the absence of equilibrium

As has been discussed many times in the literature (e.g. Drescher and do Josselin de Jong, 1972), the basic definition of Cauchy stress within a granular assembly usually relies on a volume average over a suitable Representative Volume Element, V_{RVE};

$$\langle\sigma\rangle = \frac{1}{V_{RVE}}\int_{V_{RVE}} \sigma dV \tag{1}$$

This volume average represents the average stress within a continuous domain corresponding to the discrete assembly of particles. By considering the stress within each particle separately, it is possible to express this average as;

$$\langle\sigma\rangle = \frac{1}{V_{RVE}}\sum_{P\in V_{RVE}} V^P\langle\sigma^P\rangle, \tag{2}$$

where $\langle\sigma^p\rangle$ is the volume average of stress within particle 'P', with volume V^P. With this in mind, we proceed with the rest of our analysis focused on the stress within a single particle – the above definition will allow us to relate this to bulk averages. This also applies for any time-volume or weighted time-volume averaging methods (e.g. Babic, 1997; Zhu and Yu, 2002).

Consider the particle shown in Fig. 1. This particle is subject to boundary tractions, $\{\tilde{\tau}^c\}$, via contact with other particles in the assembly, body force densities due to actions such as gravity, $\tilde{\gamma}$, and is not necessarily in equilibrium. At any point within the particle the conservation of momentum implies that;

$$\nabla\sigma + \tilde{\gamma} = \rho\ddot{\tilde{x}}, \tag{3}$$

where $\ddot{\tilde{x}}$ is the total derivative of velocity at the point in question.

Relying on the Gauss–Qstrogradsky divergence theorem, it can then be shown (e.g. Nicot et al., 2013) that the mean stress within the particle can be written;

$$\langle\sigma^p\rangle = \frac{1}{V^p}\left(\sum_c\int_{A^c}\tilde{\tau}^c\otimes\tilde{x}dA - \int_{V^p}\left(\rho\ddot{\tilde{x}} - \tilde{\gamma}\right)\otimes\tilde{x}dV\right), \tag{4}$$

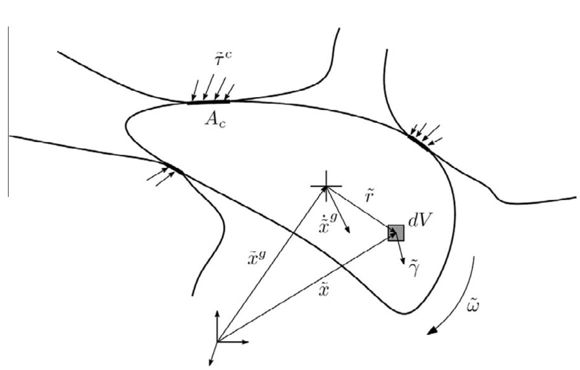

Fig. 1. A single particle forming a part of a granular assembly is subject to a number of contact traction forces, $\{\tilde{\tau}^c\}$, and a body force density, $\tilde{\gamma}$.

where the symbol $\otimes$ represents the dyadic product between vectors.

If we assume that the particle is rigid we can characterise the contact tractions as a set of discrete forces, $\left\{\tilde{f}^c = \int_{A^c}\tau^c dA\right\}$, acting at corresponding contact points $\{\tilde{x}^c\}$. Together with an assumption that the particles are homogeneous and subject to a constant body force density, Eq. (4) becomes;

$$\langle\sigma^p\rangle = \frac{1}{V^p}\sum_c\tilde{f}^c\otimes\tilde{x}^c - \frac{1}{V^p}\rho\int_{V^p}\ddot{\tilde{x}}\otimes\tilde{x}dV + \frac{1}{V^p}\tilde{\gamma}\otimes\int_{V^p}\tilde{x}dV \tag{5}$$

Thus the stress within the particle can be represented by the sum of three distinct components. For future reference, we identify them as follows;

$$\langle\sigma^p\rangle_{Lw} = \frac{1}{V^p}\sum_c\tilde{f}^c\otimes\tilde{x}^c, \tag{6}$$

is the familiar 'Love–Webber' formula (Love, 1927; Weber, 1966) describing the stress due to the contact forces;

$$\langle\sigma^P\rangle_I = -\frac{1}{V^P}\rho\int_{V^P}\ddot{\tilde{x}}\otimes\tilde{x}dV, \tag{7}$$

is an inertial component from the dynamics of the particle; and,

$$\langle\sigma^P\rangle_B = \frac{1}{V^P}\tilde{\gamma}\otimes\int_{V^P}\tilde{x}dV, \tag{8}$$

is that originating from the body forces.

The body force component can be easily simplified by expressing the position of points within the particle relative to the centre of mass, $\tilde{x} = \tilde{x}^g + \tilde{r}$, leading to;

$$\langle\sigma^P\rangle_B = \frac{1}{V^P}\tilde{\gamma}\otimes\int_{V^P}(\tilde{x}^g + \tilde{r})dV = \tilde{\gamma}\otimes\tilde{x}^g \tag{9}$$

As has been done recently by Nicot et al. (2013), we can analyse the inertial component from the perspective of the rigid body assumption by expressing the acceleration of any point within the particle as follows[1]

$$\ddot{\tilde{x}} = \ddot{\tilde{x}}^g + \dot{\tilde{\omega}}\times\tilde{r} + \tilde{\omega}\times(\tilde{\omega}\times\tilde{r}) \tag{10}$$

where $\tilde{\omega}$ is the angular velocity of the particle. Substituting this into the inertial component of stress we obtain the following;

$$\langle\sigma^P\rangle_i = -\rho\ddot{\tilde{x}}^g\otimes\tilde{x}^g - \frac{1}{V^P}\rho\int_{V^P}\left(\dot{\tilde{\omega}}\times\tilde{r}\right)\otimes\tilde{r}dV - \frac{1}{V^P}\rho\int_{V^P}(\tilde{\omega}\times(\tilde{\omega}\times\tilde{r}))\otimes\tilde{r}dV \tag{11}$$

With the aid of the vector triple product rule; $\tilde{a}\times(\tilde{b}\times\tilde{c}) = \tilde{b}(\tilde{a}\cdot\tilde{c}) - \tilde{c}(\tilde{a}\cdot\tilde{b})$ Eq. (11) can be written;

$$\langle\sigma^P\rangle_I = -\rho\ddot{\tilde{x}}^g\otimes\tilde{x}^g - \frac{1}{V^P}\rho\int_{V^P}\left(\dot{\tilde{\omega}}\times\tilde{r}\right)\otimes\tilde{r}dV - \frac{1}{V}\rho\int_{V^P}(\tilde{\omega}\cdot\tilde{r})\tilde{\omega}\otimes\tilde{r}dV + \frac{1}{V^P}\rho\int_{V^P}(\tilde{\omega}\cdot\tilde{\omega})\tilde{r}\otimes\tilde{r}dV, \tag{12}$$

or in component form;

$$\langle\sigma^p_{ij}\rangle_I = -\rho\ddot{x}^g_i x^g_j - \frac{1}{V^P}\rho\int_{V^P}\varepsilon_{ikl}\dot{\omega}_k r_l r_j dV - \frac{1}{V^P}\rho\int_{V^P}\omega_l r_l\omega_i r_j dV + \frac{1}{V^P}\rho\int_{V^P}\omega_k\omega_k r_i r_j dV, \tag{13}$$

where ε_{ijk} is the usual Levi–Civita permutation symbol.

[1] Due to the rigid body assumption, we have not explicitly written Eq. (3) in terms of convected derivatives (as has been done previously by Luding (2010)). In this instance, the effects of rotation within the material are captured by Eq. (10).