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A smoothed inverse eigenstrain method for reconstruction of the regularized residual fields

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ABSTRACT

A smoothed inverse eigenstrain method is developed for reconstruction of residual field from limited strain measurements. A framework for appropriate choice of shape functions based on the prior knowledge of expected residual distribution is presented which results in stabilized numerical behavior. The analytical method is successfully applied to three case studies where residual stresses are introduced by inelastic beam bending, laser-forming and shot peening. The well-rehearsed advantage of the proposed eigenstrain-based formulation is that it not only minimizes the deviation of measurements from its approximations but also will result in an inverse solution satisfying a full range of continuum mechanics requirements. The smoothed inverse eigenstrain approach allows suppressing fluctuations that are contrary to the physics of the problem. Furthermore, a comprehensive discussion is performed on regularity of the asymptotic solution in the Tikhonov scheme and the regularization parameter is then exactly determined utilizing Morozov discrepancy principle. Gradient iterative regularization method is also examined and shown to have an excellent convergence to the Tikhonov–Morozov regularization results.

1. Introduction

Residual stresses are generated in engineering structures as a result of a variety of manufacturing processes. So it is important to correctly quantify Residual stress field to determine the integrity and durability of an engineering structure. It is also well known that uncontrolled residual stresses are detrimental to the performance and lifetime of engineering components (Withers, 2007). The residual stresses in engineering structures are generally determined by interpretation of experimental measurements or process modeling where major limitations exist in both approaches (Jun and Korsunsky, 2010). Residual stresses can be measured using various experimental methods (Withers et al., 2008) introducing uncertainty on the residual field distribution. Although statistical methods (Faghidian, 2013; Wimpory et al., 2009) provide confidence intervals on the measured residual stress distribution but the results do not satisfy the necessary continuum mechanics requirements. Theory of inverse methods has been developed over the past decades and gained a great attention including determination of residual stress field from limited experimental measurements (Ma et al., 2012). The inverse eigenstrain method is a semi-empirical approach based on the theory of eigenstrains that combines experimental characterization in terms of residual elastic strains. The general framework of inverse problem of eigenstrain approach was introduced by Hill (1996), Cao et al. (2002) and Qian et al. (2004). The approach was then developed by Korsunsky (2005, 2006), Korsunsky et al. (2007) investigating various aspects of the framework and a least squares approach was used to determine unknown eigenstrain distributions from limited measurements of residual elastic strains. A detail description of the inverse eigenstrain approach is given by Jun and Korsunsky (2010). An alternative approach, which does not utilize assumed eigenstrain distribution, is to introduce a series of stress functions that directly solve the stress equilibrium equations together with traction free boundary conditions. In a series of publications (Farrahi et al., 2009a,b, 2010; Faghidian et al., 2012a,b) the stress function approach that does not require numerical tools such as the finite element or boundary element methods, successfully reconstruct the complete residual stress field in a variety of processes and geometries. Recently Coules et al. (2014) introduced a finite element based method for reconstruction of general three-dimensional residual stress distribution from measurements made in an incompatible region without determination of the eigenstrain distribution. Also Nedin and Vatulyan (2013) proposed analytical solution for the vibration of thin plates with non-homogeneous pre-stress fields to reconstruct residual stresses by the acoustical method.

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It is well known that the residual stresses may not be measured directly and experimental methods typically measure the distortion and hence strain in the specimen (Faghidian et al., 2012a,b). In the present study, variational inverse eigenstrain approach would be reconsidered and modified to reconstruct the residual elastic strain, stress and eigenstrain field form experimental strain measurements. The proposed smoothed inverse eigenstrain approach results in non-singular and smooth reconstructed residual fields while satisfying all of the continuum mechanics requirements. A framework for appropriate choice of non-linear shape functions based on the prior knowledge of the expected distribution is also presented in three different process of inelastic beam bending, laser-forming and shot peening. Furthermore, a comprehensive discussion is performed on the various mathematical aspect of the numerical reconstruction consisting of invertibility, uniqueness, well-posedness and convergence of the asymptotic residual field solution. The regularity of the approximated solution is also completely discussed in the Tikhonov scheme and the regularization parameter is then exactly determined utilizing Morozov discrepancy principle. Moreover since iterative regularization methods are known to be an attractive alternative to Tikhonov regularization, gradient iterative method is examined here and the results are compared to Tikhonov–Morozov regularization for the level of accuracy and the rate of convergence issues.

2. Smoothed inverse eigenstrain analysis

2.1. Residual stresses

Residual stresses are generally characterized as the stress filed supported in a continuum with a fixed reference configuration where there is no external forces and thermal gradients (Hoger, 1986). The region of interest containing the distribution of longitudinal residual stress $\sigma_{xx} = \sigma(z)$ is a plate infinitely extended in x direction with depth of h in the z direction. The plate geometry is illustrated in Fig. 1. All residual stresses and strains are assumed to be independent of x and y and only dependent on z coordinate. This is similar to the approach adopted in earlier works (Korsunsky, 2005, 2006) to examine residual stress field in beams and plates.

The residual stress field must satisfy the equilibrium equations in the absence of body forces which is described as $\sigma_{ji,j} = 0$ in plane Cartesian coordinates. Furthermore the traction free boundary conditions should also be satisfied as $\sigma_{ji}n_j = 0$ where $\boldsymbol{n}$ denotes the outward unit surface normal (Timoshenko and Goodier, 1970). A necessary condition for residual stresses is that the Cartesian components of the mean residual stress are always zero, so $\int_V \sigma_{ij} dV = 0$ where V is the volume of the continuum (Mura, 1987). The zero mean residual stress for the introduced domain reduces to,

$$\int_0^h \sigma_{xx}(z)dz = 0 \tag{1}$$

Following the stress function approach (Faghidian et al., 2012a,b), an appropriate form of the Airy stress function is given by a truncated series that satisfies the stress equilibrium equations together with the traction free boundary conditions and the mean residual stress over the entire domain is zero. The proposed Airy stress function for the domain of interest is given by,

$$\varphi(z) = z^2(z-h)^2\left[\sum_{m=1}^{M} c_m f_m(z)\right] \tag{2}$$

where c_m are unknown real coefficients yet to be determined and $f_m(z)$ is a series of shape functions to be selected based on the prior knowledge of the expected distribution of residual stress field. Also note that the shape functions of $f_m(z)$ should have at least continuous derivatives over the entire domain. Therefore, utilizing the Airy stress function would result in the final form of the longitudinal residual stress as,

$$\begin{aligned}\sigma_{xx}(z) &= \frac{\partial^2\varphi(z)}{\partial z^2} = \sum_{m=1}^{M} c_m\psi_m(z)\\ &= 2\left(h^2 - 6hz + 6z^2\right)\left[\sum_{m=1}^{M} c_m f_m(z)\right]\\ &\quad + 4z\left(h^2 - 3hz + 2z^2\right)\left[\sum_{m=1}^{M} c_m f'_m(z)\right]\\ &\quad + z^2(z-h)^2\left[\sum_{m=1}^{M} c_m f''_m(z)\right]\end{aligned} \tag{3}$$

Since the domain of interest coincides on the x–z plane, no gradient in the y direction is allowed. Hence for both conditions of the plane-stress and plane-strain, the reconstructed in-plane residual stress state is credible. Also it should be noted that the smoothed inverse eigenstrain method works independent of the constitutive material behavior and the plate bending theory assumptions. The kinematics of the deformation in terms of the strain compatibility equations would be completely discussed in Section 2.2. Nonetheless, the static equilibrium across the continuum domain expressed in terms of resultant force and moment is also guaranteed respectively as,

$$\begin{aligned}&\int_0^h \sigma_{xx}(z)dz = \int_0^h \frac{\partial^2\varphi(z)}{\partial z^2}dz = \left.\frac{\partial\varphi(z)}{\partial z}\right|_0^h = 0\\ &\int_0^h z\sigma_{xx}(z)dz = \int_0^h z\frac{\partial^2\varphi(z)}{\partial z^2}dz = z\left.\frac{\partial\varphi(z)}{\partial z}\right|_0^h - \left.\frac{\partial\varphi(z)}{\partial z}\right|_0^h = 0\end{aligned} \tag{4}$$

It will be shown in the Section 2.2 that the stress function approach, developed by the author (Faghidian et al., 2012a,b), could be appropriately modified to reconstruct the residual elastic strain and eigenstrain field form of limited strain measurements.

2.2. Residual elastic strains

The direct problem of eigenstrain is known as the problem of determination of residual stresses resulting from a known eigenstrain distribution utilizing generalized Hooke's law (Korsunsky, 2009). The residual stress field is well known to depend on the internal incompatibility within the continuum. The distribution of incompatible strain field in such a continuum is known as the eigenstrain distribution, and it is generally assumed that total strain tensor ε_{ij} can be expressed as the sum of the elastic e_{ij} and eigenstrain terms ε^*_{ij} as (Mura, 1987),

$$\varepsilon_{ij} = e_{ij} + \varepsilon^*_{ij} \tag{5}$$

According to Korsunsky (2006), in the absence of external loading being applied, elastic strain presents an illustration of residual elastic strain (R.E.S.), such as that measured in a diffraction experiments. Also Kinematic analysis of continuous deformation requires the strain field to be compatible. Hence, the total strain

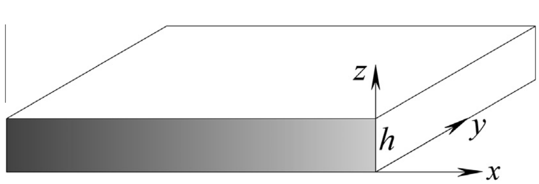

Fig. 1. Illustration of the problem geometry.