International Journal of Solids and Structures 51 (2014) 4452–4461

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Dispersion and localisation in structured Rayleigh beams

A. Piccolroaz [a,*], A.B. Movchan [b]

[a] Dipartimento di Ingegneria Civile, Ambientale e Meccanica, Università di Trento, Italy
[b] Department of Mathematical Sciences, University of Liverpool, UK

ARTICLE INFO

Article history:
Received 25 August 2014
Available online 2 October 2014

Keywords:
Rayleigh beam
Rotational inertia
Dispersive waves
Localised wave forms
Quasi-periodic Green's functions

ABSTRACT

This paper brings a comparative analysis between dynamic models of couple-stress elastic materials and structured Rayleigh beams on a Winkler foundation. Although physical phenomena have different physical origins, the underlying equations appear to be similar, and hence mathematical models have a lot in common. In the present work, our main focus is on the analysis of dispersive waves, band-gaps and localised waveforms in structured Rayleigh beams. The Rayleigh beam theory includes the effects of rotational inertia which are neglected in the Euler–Bernoulli beam theory. This makes the approach applicable to higher frequency regimes. Special attention is given to waves in pre-stressed Rayleigh beams on elastic foundations.

1. Introduction and analogy between waves in Rayleigh beams and couple-stress elastic materials

Bloch waves in structured media have received a significant attention in models of photonic and phononic crystals, that encompass problems of electro-magnetism, optics, acoustics and more recently elasticity (Yablonovitch, 1987; John, 1987; Kushwaha et al., 1993; Sigalas and Economou, 1993). An important feature of elastic waves, even in an isotropic case, is the presence of two types of waves linked to dilatation and shear, respectively.

More recently, there was a significant interest generated by studies of micropolar media and couple-stress materials (see, for example, Morini et al. (2013), Engelbrecht et al. (2013), Zisis et al. (2014), Gourgiotis and Piccolroaz (2014)). In particular, the Mindlin's approach (Mindlin, 1964) leads to additional higher-order derivatives in the governing equations. Engelbrecht et al. (2005) follow Mindlin's interpretation of a micro-structure as a polycrystal, whose micro-elements are taken as deformable cells. In the limit when the cells are rigid, their approach would lead to the Cosserat model. It was noted that the higher-order model leads to the novel dispersion properties of waves supported by such a micro-structured medium. We also note a formal equivalence between constrained Cosserat model and the couple-stress model of structured media (see, for example, Koiter (1964), Mindlin and Tiersten (1962)), and a further analogy with the theory of Rayleigh beams which accounts for rotational inertia, which will be discussed in the text below.

When rotational inertia is neglected in analysis of flexural waves, this is fully appropriate in the long wave approximations and is well adopted to the Euler–Bernoulli beam theory. Dispersion and filtering of elastic waves in structured prestressed Euler–Bernoulli beams were considered in Gei et al. (2009), which includes analysis of band-gap shift, defect-induced annihilation and localised modes. It is also known that the couple-stress effects are neglected in the classical models of linear elasticity.

It appears that there is a mathematical underlying framework, which applies both to the couple-stress approach in elasticity as well as flexural waves in beam models accounting for the effects of rotational inertia.

The Rayleigh beam theory is used here to account for the rotary motion of beam elements. This approach also allows for the description of flexural waves at high frequency ranges (Graff, 1991). In the Rayleigh beam theory, the assumptions regarding the geometry of the deformation and the material properties remain unchanged, so that the rotation ϕ of the cross sections is not an independent parameter, but it is *constrained* to the transverse displacement v by the relation $\phi = v'$. However, in writing the equations of motion, both the translational inertia and the rotational inertia of beam elements are taken into account, so that

$$V' = \beta v + \rho A \ddot{v}, \quad M' = V - P v' - \rho I \ddot{v}', \tag{1}$$

where V is the internal shear force, M the internal bending moment, β the stiffness of a Winkler type elastic foundation, P the prestress, ρ the mass density, A the area of the cross-section, and I the area moment of inertia of the cross-section.

Combining the equations of motion with the *constitutive* equation $M = -EI v''$, where EI is the bending stiffness, we obtain

* Corresponding author. Tel.: +39 0461 282583.
E-mail address: roaz@ing.unitn.it (A. Piccolroaz).

http://dx.doi.org/10.1016/j.ijsolstr.2014.09.016

the governing equation for the transverse motion v of a homogeneous prestressed Rayleigh beam resting on an elastic foundation of the Winkler type

$$EIv'''' - Pv'' + \rho A\ddot{v} - \rho I\ddot{v}'' + \beta v = 0. \tag{2}$$

We note that this physical problem has a formal mathematical analogy with the problem of shear wave propagation in couple-stress elastic materials. In fact, the dynamic equation governing the anti-plane motion in a couple-stress elastic material is given by Mishuris et al. (2012)

$$G\ell^2\Delta^2 w - 2G\Delta w + 2\rho\ddot{w} - \frac{J}{2}\Delta\ddot{w} = 0, \tag{3}$$

where w is the out-of-plane displacement; the shear modulus G and the mass density ρ are the classical *macroscopic* material parameters, whilst the characteristic length ℓ and the micro-rotational inertia J are the generalised parameters connected with the *microstructure*. Clearly, Eq. (3) is the two-dimensional analogue of the Rayleigh beam Eq. (2) with $\beta = 0$.

Both Eqs. (2) and (3) are not classical wave equations, because of the fourth-order terms. In the case of the Rayleigh beam, the coefficients near fourth-order terms include the bending stiffness EI and the rotational inertia ρI, showing that they are essentially linked to the geometrical properties of the structural element, in particular the area moment of inertia of the cross-section. On the other hand, in the case of a couple-stress material, the fourth-order terms are proportional to the microstructural parameters $G\ell^2$ and J. In both cases, these higher-order terms are responsible for the dispersive character of wave propagation.

In several studies on wave propagation in generalized continua (Mindlin, 1964; Gourgiotis et al., 2013; Morini et al., 2014), it has been shown that elastic waves can be made non-dispersive for special values of the rotational inertia. In particular, for shear waves in couple stress materials described by Eq. (3), this value is given by $J = 2\rho\ell^2$ (Mishuris et al., 2012). By analogy, we obtain that flexural waves in a Rayleigh beam are non-dispersive for a special value of the prestress, namely $P = EA$.

The key features characterising a dynamic response of a Rayleigh beam on the Winkler foundation are linked to the presence of exponential and quasi-polynomial terms in the representation of flexural displacements. In particular, exponential terms account for the high gradient regions, and of course they can be seen in the representation of dynamic Green's functions. For the purpose of illustration, we show in Fig. 1 the graphs of flexural displacements $v(x)$ produced by the time-harmonic point force applied to the Rayleigh beam at the origin. These represent the special resonant cases when the solution has a linear growth at infinity; the oscillation in Fig. 1b is produced as a result of a compressive prestress.

The structure of the paper is as follows. Section 2 presents a detailed account of dispersion properties of flexural waves in a homogeneous prestressed Rayleigh beam supported by a Winkler foundation. In Section 3 we consider a periodic multi-phase medium, with Bloch waves representing quasi-periodic solutions of the equation of motion. Analysis of band gaps and filtering properties is included in this section. Section 4 includes analytical closed form representations for dynamic Green's functions and their derivatives (concentrated couples), that describe the full range of admissible localised and propagating wave forms. Finally, in Section 5, we construct a special class of so-called quasi-periodic Green's functions, required to study a dynamic response of periodic systems of masses placed along the Rayleigh beam; this also includes periodic system of bodies with given rotational inertia. The coupling between the rotational action and transverse motion of masses appears to be important, which is discussed in that section. The analysis is generic and our systematic study is applicable to flexural systems such as plates and shells, especially for the cases where rotational inertia appears to provide a significant contribution.

2. Dispersion properties of a homogeneous prestressed Rayleigh beam on an elastic foundation

The dispersion relation for a homogeneous prestressed Rayleigh beam, governed by (2), is obtained by assuming that a sinusoidal signal $v = \xi \exp i(kx - \omega t)$ is propagating in the beam, which gives

$$\omega = \sqrt{\frac{k^2r^2(k^2r^2 + \bar{P}) + \bar{B}}{R(k^2r^2 + 1)}}, \tag{4}$$

where r is the radius of inertia of the beam cross-section

$$r = \sqrt{\frac{I}{A}}, \tag{5}$$

$\bar{P}$ and $\bar{B}$ are the dimensionless prestress and foundation stiffness, respectively,

$$\bar{P} = \frac{Pr^2}{EI} = \frac{P}{EA}, \quad \bar{B} = \frac{\beta r^4}{EI} = \frac{\beta I}{EA^2} \tag{6}$$

and R is the normalised inertia term, having the dimension of a squared time

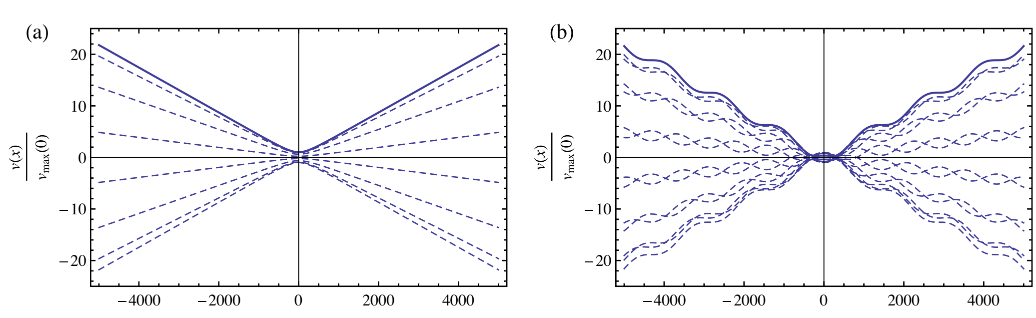

Fig. 1. Flexural vibrations produced by a time-harmonic point force applied at the origin to a Rayleigh beam on the Winkler foundation in the special resonant cases when the solutions have a linear growth at infinity (see Section 4, formulae (40) and (41)). We plot several profiles captured at different times within the period of the oscillatory force. A beam with circular cross section is assumed having the following properties: Young modulus 210 GPa, mass density 7.85·10^3 kg/m^3, foundation stiffness 2.64 MPa, radius of the cross Section 0.01 m. The angular frequency is 1034.4 s^{-1}. Transversal displacements $v(x)$ are normalised with respect to the maximum value at the origin $v_{max}(0)$, whereas the longitudinal position x is normalised with respect to the radius of the cross-section a. (a) Tensile prestress equal to 1.32 kN. (b) Compressive prestress equal to −1.19 kN.