International Journal of Solids and Structures 51 (2014) 4485–4491

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Analytical representation of the non-square-root singular stress field at a finite angle sharp notch

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ARTICLE INFO

Article history:
Received 24 February 2014
Received in revised form 22 July 2014
Available online 6 September 2014

Keywords:
Notch
Generalized stress intensity factor
Fracture

ABSTRACT

The stress field near the tip of a finite angle sharp notch is singular. However, unlike a crack, the order of the singularity at the notch tip is less than one-half. Under tensile loading, such a singularity is characterized by a generalized stress intensity factor which is analogous to the mode I stress intensity factor used in fracture mechanics, but which has order less than one-half. By using a cohesive zone model for a notional crack emanating from the notch tip, we relate the critical value of the generalized stress intensity factor to the fracture toughness. The results show that this relation depends not only on the notch angle, but also on the maximum stress of the cohesive zone model. As expected the dependence on that maximum stress vanishes as the notch angle approaches zero. The results of this analysis compare very well with a numerical (finite element) analysis in the literature. For mixed-mode loading the limits of applicability of using a mode I failure criterion are explored.

1. Introduction

Consider the stress field in the vicinity of a finite angle notch in a flat plate (Fig. 1) under symmetric (e.g. tensile) loading. It is known that for notch angles $\beta < \pi$, the stress field is infinite in the limit as $r \to 0$ (Williams, 1952). However it is only for a crack (i.e. $\beta = 0$) that the well-known square-root singularity exists. In such cases it is the *stress intensity factor* (K_I for mode I, i.e. tensile loading) which governs the local stress field in the immediate vicinity of the crack (e.g. Ewalds and Wanhill, 1984). In particular the normal stress acting on the symmetry plane ($\theta = 0$) is given by

$$\tau_{\theta\theta}(r,0) = \frac{K_I}{\sqrt{2\pi}} r^{-1/2}, \quad \text{as} \quad r \to 0 \tag{1}$$

It is, of course, also known that the infinite stresses predicted by linear elasticity cannot exist. Nonetheless knowledge of the stress intensity factor is what is needed to predict crack propagation in the context of linear elastic fracture mechanics. If $K_I \geqslant K_{IC}$ the crack will propagate; otherwise it will not. The quantity K_{IC} is known as the *critical stress intensity factor*, or *fracture toughness* and characterizes the resistance of the material to fracture.

Rather than using the stress intensity factor, another point-of-view is that given by the energy release rate (G) and the related J-integral, i.e.

$$G = J = K^2/E^* \tag{2}$$

where $E^* = E$ for plane stress and $E^* = E/(1-\nu^2)$ for plane strain, E is Young's modulus, and ν is Poisson's ratio. Thus crack growth will occur when G or equivalently J reaches a critical value equal to K_{IC}^2/E^*.

Suppose now that the notch angle is not zero. The stresses are still singular for $0 < \beta < \pi$, but the order of the singularity is given by λ, where $0 < \lambda < \frac{1}{2}$. The normal stress acting on the symmetry plane ($\theta = 0$) is then

$$\tau_{\theta\theta}(r,0) = Qr^{-\lambda}, \quad \text{as} \quad r \to 0 \tag{3}$$

where Q is referred to as the *generalized stress intensity factor* (e.g. Malyshev and Salganik, 1965). Thus for a crack, Q as defined by Eq. (3), differs from K_I by a factor of $\sqrt{2\pi}$. More importantly, because $\lambda \neq \frac{1}{2}$ the road to predicting fracture is not at all clear.

Problems involving adhesive joints are also characterized by non-square-root singularities. Thus there have been numerous papers addressing the singular stress state which exists at the bi-material interface of two wedges in an adhesive joint (Malyshev and Salganik, 1965; Gradin, 1982; Groth, 1985, 1988; Munz and Yang, 1993; Tan and Meguid, 1997; Reedy, 2000; Reedy and Guess, 1997). These problems are further complicated by mixed mode loading (tension and shear). Furthermore failure can occur in an intervening adhesive material. Hence most of this body of work is devoted to taking the results of specific adhesive

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http://dx.doi.org/10.1016/j.ijsolstr.2014.08.024
0020-7683/

experiments and generalizing to other geometries and loading conditions.

Experiments have been conducted for three-point bending of notched PMMA beams for different notch angles. Carpinteri (1987) compared the critical force required for failure for each notch angle to the corresponding critical force for propagation of a crack. Dunn et al. (1997a) presented experimental results for four different notch depths. By using finite element analysis, they were able to show that for *each* wedge angle there was a consistent value of the generalized stress intensity factor to initiate failure for different notch depths. In neither paper was a basis given for relating the critical value of the *generalized* stress intensity factor (Q_C) to the fracture toughness (K_{IC}). A finite element analysis of edge and center notches, without experimental results, was conducted by Dunn et al. (1997b).

A brittle fracture criterion for structures with sharp notches was proposed by Seweryn (1994) based on the fracture of a crack at the tip of the notch. A fracture criterion for sharp notched samples in mode I was given by Gómez and Elices (2003). They related the critical value of the generalized stress intensity factor to that of a crack using a finite element analysis. Very good agreement with experimental results from a variety of materials was obtained. Experimental results for mixed-mode loading are given by Gómez et al. (2009) in which a simple fracture criterion was proposed based on the dominance of local mode I. Ayatollahi et al. (2011) also present experimental results along with a failure criterion based upon the maximum tangential stress suggested by Erdogan and Sih (1963) for sharp cracks.

The relationship between the singular stress field at a sharp notch, and the case of a rounded notch has also been investigated. Lazzarin and Tovo (1998) investigated the stress field near welds in terms of generalized stress intensity factors. The range of validity and the limits of applicability of these results in the presence of weld toe radii were discussed. Dini and Hills (2004) showed how the notch radius affects the singular field. They also established a lower bound for loads at which the plastic zone is characterized by the singular field (Dini and Hills, 2006). The effect of the finite notch radius on fracture was investigated by Gómez and Elices (2004) using a cohesive zone model. The results from the theory compared well with experimental results published in the literature for a variety of materials.

The contributions of mode I and mode II stress fields were quantified by Lazzarin and Zambardi (2001) using a control volume concept. The link between the radius of this volume and the dimension of the cohesive zone model became evident. The singular state of stress near the tip of a notch has also been investigated by Hills and Dini (2011) for mixed-mode loading. They observed that because the order of the mode I singularity was greater than that of the mode II, then as the notch tip is approached mode I will eventually dominate the purely elastic solution. However when taking into account a finite length process zone, the relevant question is which singularity dominates at the beginning of the process zone. Their results showed that it could be mode I, mixed-mode, or mode II depending on the loading and the material properties.In this paper, we use an analytical approach to revisit the problem studied with finite elements by Gómez and Elices (2003). Namely suppose that for a given material we know the critical mode I stress intensity factor (K_{IC}), and for a certain problem (e.g. tension or bending of a plate with a finite angle notch) the order of the singularity and the mode I generalized stress intensity factor are calculated. Can we use these values to determine whether or not fracture at the notch will occur? We determine the relation between the critical generalized stress intensity factor (Q_C) and the fracture toughness (K_{IC}). The results agree very well with the finite element results of Gómez and Elices (2003). We then explore the limits of applicability of this mode I failure criterion under mixed-mode loading.

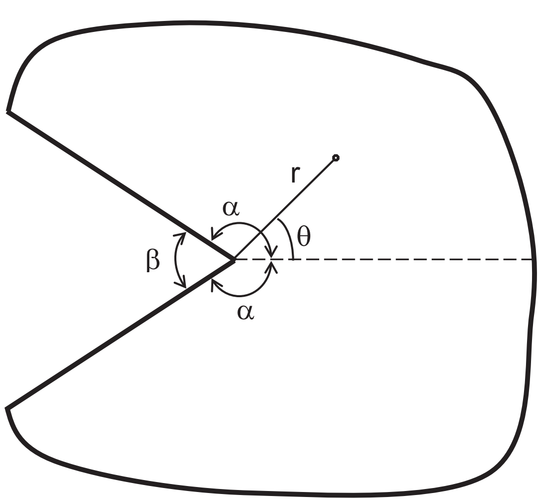

Fig. 1. The immediate vicinity of a wedge of angle 2α.

2. Asymptotic analysis for solid wedge (2α)

Consider a finite angle notch in a flat elastic plate with coordinate system (r,θ) as shown in Fig. 1. The order of the singularity can be obtained using the asymptotic approach of Williams (1952). We write the boundary conditions along the two free surfaces ($\theta = \pm\alpha$) as

$$\tau_{\theta\theta}(r,\alpha) = 0, \quad \tau_{r\theta}(r,\alpha) = 0, \quad \tau_{\theta\theta}(r,-\alpha) = 0, \quad \tau_{r\theta}(r,-\alpha) = 0. \quad (4)$$

These conditions are applied to the stress fields which are summarized by Comninou (1977) in a convenient form as

$$\begin{aligned} \tau_{r\theta}(r,\theta) &= [A\lambda \sin\lambda\theta - B\lambda\cos\lambda\theta + C\sin(2-\lambda)\theta \\ &\quad + D\cos(2-\lambda)\theta](1-\lambda)r^{-\lambda} \\ \tau_{\theta\theta}(r,\theta) &= [A(2-\lambda)\cos\lambda\theta + B(2-\lambda)\sin\lambda\theta + C\cos(2-\lambda)\theta \\ &\quad - D\sin(2-\lambda)\theta](1-\lambda)r^{-\lambda} \end{aligned} \quad (5)$$

Following the procedure of Williams (1952), the result of applying the boundary conditions (4) to Eq. (5) is a system of four homogeneous linear algebraic equations, the determinant of which must vanish for non-trivial solutions of A, B, C, and D. Setting the determinant to zero yields

$$\pm\sin\ 2(1-\lambda)\alpha + (1-\lambda)\sin\ 2\alpha = 0 \quad (6)$$

in which the plus and minus signs in the first term correspond to symmetric (mode I) and antisymmetric (mode II) solutions respectively.

It is noted that for $90^\circ < \alpha \leqslant 180^\circ$ the numerical solution of Eq. (6) for mode I gives one real root in the range $0 < \lambda < 1$ and no complex roots. Roots greater than unity are not allowed based upon energy considerations, whereas negative values of λ lead to bounded stresses as $r \to 0$ and consequently are not of interest in this study. Similarly mode II solutions of Eq. (6) exist only for $128.7^\circ < \alpha \leqslant 180^\circ$.

The variation of the orders of the singularities (λ) with the notch angle ($\beta = 360^\circ - 2\alpha$) is shown in Fig. 2 for mode I and for mode II. The order of the singularity decreases from one-half when $\beta = 0$ to zero for mode I when $\beta = 180^\circ$ and for mode II when $\beta = 102.6^\circ$. However even for a large notch angle of 90°, the order of the singularity is reasonably close to one-half (approximately 0.4552) for mode I but small for mode II (about 0.0915). It is noted that the results shown in Fig. 2 are independent of material properties