International Journal of Solids and Structures 51 (2014) 4531–4543

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

On the effect of interactions of inhomogeneities on the overall elastic and conductive properties

Igor Sevostianov [a,*], Mark Kachanov [b], Borys Drach [a]

[a] Department of Mechanical and Aerospace Engineering, New Mexico State University, PO Box 30001, Las Cruces, NM 88003, USA
[b] Department of Mechanical Engineering, Tufts University, Medford, MA 02155, USA

ARTICLE INFO

Article history:
Received 27 May 2014
Received in revised form 3 August 2014
Available online 16 September 2014

Keywords:
Interaction
Inhomogeneity
Effective properties
Cross-property connection

ABSTRACT

A simple method of estimating the effect of inhomogeneity interactions on the overall properties (elastic and conductive) is developed. It is formulated in terms of property contribution tensors that give the contribution of an inhomogeneity to the overall properties. The method can be viewed as further development of the approach of Rodin and Hwang (1991) and Rodin (1993) that generalized the method of analysis of crack interactions (Kachanov, 1987) to inhomogeneities. We also extend the method to the conductive properties. Considering the effect of interactions on the property contribution tensors on the example of pores we find that this effect is generally moderate, at most (even when pores touch one another) – in contrast with the effect on local fields. On example of two spheres, we compare the interaction effects on the elastic and the conductive properties, and discuss the impact of interactions on the cross-property connections.

1. Introduction

Interactions of inhomogeneities – in the theories of elasticity and conductivity – is a problem that has substantial history. The emphasis seems to have been on *local* fields (in the context of cracks – on stress intensity factors, SIFs). The present work has a more specific focus – on the effect of interactions on *volume average* (effective) properties. Of interest, then, are the average over inhomogeneities quantities (strains, temperature gradients). To this end, a simple method of analysis is proposed whereby the said averages are found by interrelating them by linear algebraic equations. The method is formulated for both elasticity and conductivity problems.

Yet another goal is to compare interaction effects in the elasticity and conductivity problems. The comparison is of importance for the explicit cross-property connections established by Sevostianov and Kachanov (2002) (see also their review of 2009). They have been derived under the assumption that the inhomogeneities do not interact. Experimental data indicate, however, that they remain valid at substantial concentrations of inhomogeneities. A comparative analysis of the interaction effects in the two problems clarifies the reason why this is the case.

The effect of interactions on *local* elastic fields has been first analyzed, probably, in the work of Sternberg and Sadowski (1952) where the axisymmetric problem of two spherical pores of equal size was analyzed using spherical harmonics. Chen and Acrivos (1978) constructed a solution for two spherical inhomogeneities of equal size in the form of multipole expansions; they estimated that their analysis was accurate at distances between spheres larger than their radius. Rodin and Hwang (1991) showed that this estimate is overly conservative and the method can actually be applied at distances larger than 0.25 of the radius. Tagliavia et al. (2011) used the approach of Chen and Acrivos to calculate effective properties of synthetic foams.

A more general approach to the problem of two inhomogeneities was developed by Moschovidis and Mura (1975). Their approximate solution is based on the theorem on polynomial conservation (Kunin and Sosnina, 1971; see, also, Asaro and Barnett (1975)). In order to reduce the boundary-value problem to a system of linear algebraic equations, the field acting on each inhomogeneity was represented by Taylor's series. This approach can be extended to N inhomogeneities, as mentioned by Mura (1987). Johnson et al. (1980) stated that a more accurate solution can be obtained by the Taylor expansion centered at the point where stresses are to be calculated. This statement was numerically verified by Benedict et al. (2006). Zhou et al. (2011) extended the Taylor series approach to inhomogeneities of arbitrary shapes.

* Corresponding author.
E-mail addresses: igor@nmsu.edu (I. Sevostianov), mark.kachanov@tufts.edu (M. Kachanov).

http://dx.doi.org/10.1016/j.ijsolstr.2014.08.029
0020-7683/

Nomenclature (in alphabetic order)

Greek letters

Γ_{ijkl}	stress concentration tensor
ε_{ij}	strain tensor
Θ_{ijkl} and Θ_{ij}	strain concentration tensor and temperature gradient tensor, respectively
$\Lambda^{ps}_{ijkl} \equiv \langle D^{p}_{ijkl}(\boldsymbol{x})\rangle_{\Omega_s}$ and $\Lambda^{ps}_{ij} \equiv \langle D^{p}_{ij}(\boldsymbol{x})\rangle_{\Omega_s}$	transmission tensors in the context of elasticity and conductivity problems, respectively
ν	Poisson's ratio of isotropic linear elastic material
σ_{kl}	stress tensor
ϕ	volume fraction of inhomogeneities

Latin letters

C_{ijkl}	stiffness tensor of a material
$D_{mnij}(\boldsymbol{x})$	external strain field generated by an isolated inhomogeneity experiencing uniform strains with components of unit magnitude
$D_{ij}(\boldsymbol{x})$	external temperature gradient field generated by an isolated inhomogeneity experiencing uniform temperature gradient with components of unit magnitude
G_{ij} and G	tensor and scalar Green's functions for elasticity and conductivity problems, respectively
H_{ijkl}	compliance contribution tensor
$J_{ijkl} = (\delta_{ik}\delta_{lj}+\delta_{il}\delta_{kj})/2$	fourth rank unit tensor with components
K_{ij}	conductivity contribution tensor
k_{ij}	conductivity tensor of a material
N_{ijkl}	stiffness contribution tensor
P_{ijkl} and P_{ij}	hill tensors of an inhomogeneity for elasticity and conductivity problems respectively
q_i	heat flux vector
R_{ij}	resistivity contribution tensor
r_{ij}	resistivity tensor of a material
S_{ijkl}	compliance tensor of a material
s_{ijkl}	Eshelby's tensor of an inhomogeneity
T	temperature

Superscripts

"∞"	remotely applied field
"0"	matrix material
"1"	material of the inhomogeneity
"in"	field inside an inhomogeneity

Willis and Acton (1976) considered the effective elastic properties for weakly interacting spherical inhomogeneities through calculation of the square-of-concentration term, using the far-field solution. Comparison with numerical simulations shows that the results are accurate at spacing larger than 0.5 of the radius of the spheres (Rodin and Hwang, 1991). Similar approach has been proposed by Chen and Acrivos (1978b) who constructed (using the approach similar to the one of Jeffrey (1973) in the context of viscosity of suspensions) a solution for terms up to the square of concentration.

Kushch (1996) considered the problem of N parallel spheroidal inhomogeneities, in the context of finding full elastic fields. His approach – a version of the multipole expansion – reduces the problem to an infinite system of linear algebraic equations; the procedure converges reasonably fast. Kushch and Sevostianov (2004) extended this approach to a material with transversely-isotropic phases.

Schjodt-Thomsen and Pyrz (2005) considered a cubic arrangement of spherical inhomogeneities and used numerical integration in order to reduce a system of coupled singular integral equations to a set of algebraic equations for eigenstrains, to be solved numerically. They constructed both the local fields and the effective elastic properties.

In a narrower context of cracks, the interaction problem was first addressed by Barenblatt (1962) and Erdogan (1962) who considered a 2-D arrangement of two collinear cracks; this was followed by a large number of results (mostly numerical) on various crack arrangements that have been summarized in several handbooks of stress intensity factors (see, for example, Murakami (1987)). In the 2-D case, a methodology of dealing with arbitrary arrangements of interacting cracks using polynomial expansion of tractions on cracks and finding polynomial coefficients from a large system of algebraic equations was developed by Gross (1982); it was later called "the method of pseudo-tractions" by Horii and Nemat-Nasser (1985). This method experiences difficulties for closely-spaced cracks, as well as in 3-D geometries. A simpler method that applies to both 2-D and 3-D geometries (and is practically exact in 3-D cases) was developed by Kachanov (1985, 1987); in this method, the effect of crack A on crack B is found by assuming that crack A is loaded by uniform tractions, i.e. the effect on crack B of traction non-uniformities on crack A is neglected.

Rodin and Hwang (1991) extended the latter method to interacting spherical inhomogeneities (note that their analysis can be generalized to ellipsoids in a straightforward way). Since their primary focus was on local fields, a combination of analytical results with FEM was used (the authors called their approach a "hybrid" one). As far as the effective properties are concerned, their conclusion was that "the potential energy release which governs the overall response of composite materials is almost unaffected by interactions". Rodin (1993) used this method to calculate effective properties of a material containing infinite number of spherical inhomogeneities. To this end, he considered the limit when the number of inhomogeneities and the volume occupied by the composite material simultaneously tend to infinity.

In the context of the *conductivity* problem, Jeffery (1912) considered the arrangement of two spheres using the bi-spherical coordinates and producing solution in the form of infinite series in Legendre polynomials. As shown by Chowdhury and Christov (2010), the convergence of this series is very fast (exponential). Christov (1985) applied the bi-spherical coordinates to the heat conduction problem involving two spherical inhomogeneities under constant gradient of applied field but no numerical results were presented in his work. In the asymptotics of widely spaced inhomogeneities in the conductivity problem, a solution for two spheres was given in the earlier work of Hicks (1879), by expanding the solution in spherical harmonics around two poles – centers of the spheres (this method was later called "twin-pole expansion" by Jeffrey (1973)). Its advantage is that the integrals involved (in computation of the overall transport coefficients are easy to evaluate. For this reason, Jeffrey (1973) suggested, in the context of the theory of fluid suspensions, that the twin-pole expansion is superior to treatment in the bi-spherical coordinates. As noted by Chowdhury and Christov (2010), this claim is not obvious since the twin-pole expansion actually involves two approximations: (1) truncation of the Legendre series, and (2) coefficients of the series that depend on the radial coordinate, cannot generally be found in closed form and are sought in the asymptotics of small