International Journal of Solids and Structures 51 (2014) 3823–3843

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Solution of general integral equations of micromechanics of heterogeneous materials

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ARTICLE INFO

Article history:
Received 27 November 2013
Received in revised form 14 April 2014
Available online 12 June 2014

Keywords:
Microstructures
Inhomogeneous material
Thermoelastic material
Interface effects
Integral equations

ABSTRACT

One considers a linear composite materials (CM), which consists of a homogeneous matrix containing a random set of heterogeneities. An operator form of solution of the general integral equation (GIE) for the general cases of local and nonlocal problems, static and wave motion phenomena for composite materials with random (statistically homogeneous and inhomogeneous, so-called graded) structures containing coated or uncoated inclusions of any shape and orientation with perfect and imperfect interfaces and subjected to any number of coupled or uncoupled, homogeneous or inhomogeneous external fields of different physical nature. The GIE, connecting the driving fields and fluxes in a point being considered and the fields in the surrounding points, are obtained for the random fields of heterogeneities in the infinite media. Estimations of the effective properties and both the first and second statistical moments of fields in the constituents of CMs are presented in a general form of perturbations introduced by the heterogeneities and taking into account a possible imperfection of interface conditions. The solution methods of GIEs are obtained without any auxiliary assumptions such as the effective field hypothesis (EFH), which is implicitly exploited in the known methods of micromechanics. Some particular cases, asymptotic representations, and simplifications of proposed methods are presented for the particular constitutive equations such as linear thermoelastic cases with the perfect and imperfect interfaces, conductivity problem, problems for piezoelectric and other coupled phenomena, composites with nonlocal elastic properties of constituents, and the wave propagation in composites with electromagnetic, optic and mechanical responses.

1. Introduction

The prediction of the behavior of composite materials in terms of the properties of constituents and their microstructure is a central problem of micromechanics, which is evidently reduced to the estimation of fields in the constituents. Appropriate, but by no means exhaustive, references for the estimation of effective elastic moduli of statistically homogeneous media are provided by the reviews Shermergor (1977), Mura (1987), Nemat-Nasser and Hori (1993), Torquato (2002), Milton (2002), Buryachenko (2007a), Li and Wang (2008), Kanaun and Levin (2008) and Dvorak (2013). It appears today that variants of the effective medium method (EMM, Kröner, 1958; Hill, 1965) and the Mori–Tanaka method (MTM, Mori and Tanaka, 1973; Benveniste, 1987; Weng, 1990) are the most popular and widely used methods. Recently a new method has become known in the literature, namely the multiparticle effective field method (MEFM) that was put forward and developed by the author (see for references Buryachenko, 2007a). The MEFM is based on the theory of functions of random variables and Greens functions. Within this method one constructs a hierarchy of statistical moment equations for conditional averages of the stresses in the inclusions. The hierarchy is then cut by introducing the notion of an effective field according to which each heterogeneity is located inside a homogeneous so-called effective field. This way the interaction of different inclusions is taken into account. Thus, the MEFM does not make use of a number of hypotheses which form the basis of the traditional one-particle methods.

It is interesting that there are known the counterparts of the mentioned methods applied to CMs with another constitutive laws. Except for notations, these methods coincide with the corresponding methods of linear thermoelasticity. In light of the analogy mentioned, the general operator representation of known methods for different microinhomogeneous structures is of profound importance in both practical and theoretical sense. The current

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http://dx.doi.org/10.1016/j.ijsolstr.2014.06.008
0020-7683/

paper is dedicated to generalization of the mentioned results to its operator form for the general cases of local and nonlocal problems, static and wave motion phenomena for composite materials with the random (statistically homogeneous and inhomogeneous, so-called graded) structures containing coated or uncoated inclusions of any shape and orientation with the perfect and imperfect interfaces and subjected to any number coupled or uncoupled, homogeneous or inhomogeneous external fields of different physical nature. Particular simplified and asymptotic cases of these methods are considered and qualitatively compared with the known methods for the different specific constitutive equations.

The sketch of micromechanics of random structures can be subdivided on a few steps. At first the so-called general integral equation (GIE) should be proposed. The GIE is the exact integral equation connecting the random fields at the point being considered and the surrounding points. There is a very long and dramatic history of development of GIE which goes back to Rayleigh (1892). For the linear elasticity, Buryachenko (2001, 2007a) presented comprehensive review of the history of the classical GIE while Buryachenko (2010a,b) proposed the new GIEs forming a new background of micromechanics. The next step is formed by so-called methods of micromechanics (e.g., MTM, EMM, or MEFM) which are in fact the approximate solutions of these GIEs. It is worthy of note (see Buryachenko, 2013; Buryachenko, 2014c) that some popular approaches (e.g., EMM and MTM) are related with GIEs (even if this term is not used) at least in a sense that these approaches constitute the methods of solutions of the GIEs. In its turn, the methods of micromechanics are based on the solutions for one (or a few) inclusion inside an infinite matrix subjected to the homogeneous remote field. After substitution of the last particular solutions into GIEs, the obtained equations are solved by the use of one or another closing assumption distinguished for the different methods. In the current paper, the presented sketch of micromechanics is generalized to the operator form for the general operator case of constitutive equations.

The outline of the paper is as follows. In Section 2 we present the statistical description of the composite microstructure and the basic field equations in the general operator form of linear constitutive equations covering nonlocal and coupled phenomena. In Section 3 the new operator form of GIE connecting the driving fields and fluxes in a point being considered and the fields in the surrounding points, are presented in terms of perturbators introduced by a finite numbers of heterogeneities in the infinite media. In Section 4 the operator form of GIEs is obtained for the effective fields acting on each pair of heterogeneities. In Section 5 some particular cases, asymptotic representations, and simplifications obtained in the framework of the popular micromechanical hypotheses and concepts are considered in parallel with analyses of its connection with the known methods (such as, e.g., MTM and MEFM). Section 6 is dedicated to obtaining of the general operator form of the method of integral equations for estimation of the second moments of fields in the phases. In Section 7 one considers the solutions for composites with the particular constitutive equations such as linear thermoelastic cases with the perfect and imperfect interfaces, conductivity problem, problems for piezoelectric and other coupled phenomena, composites with nonlocal elastic properties of constituents, and the wave propagation in composites with electromagnetic, optic and mechanical responses. For lack of space, at the consideration of the particular methods of solutions of GIEs in Section 7, the readers are referred only to the references where these solutions were already analyzed and where additional references (with the corresponding numerical and experimental data) can be found.

The current paper is dedicated to generalization of the previous results (Buryachenko, 2007b, 2010a,c, 2011a,c; Buryachenko and Brun, 2011, 2012a,b, 2013) to its operator form solution for the general cases of local and nonlocal problems, static and wave motion phenomena for composite materials with statistically inhomogeneous structures containing both the coated or uncoated inclusions of any shape and orientation with perfect and imperfect interfaces and subjected to any number coupled or uncoupled, homogeneous or inhomogeneous external fields of different physical nature.

2. Preliminaries

2.1. Statistical description of the composite microstructure

Let a full space R^d with a space dimensionality d ($d=2$ and $d=3$ for 2-D and 3-D problems, respectively) contains a homogeneous matrix $v^{(0)}$ and, in general, a statistically inhomogeneous set $X=(v_i)$ of heterogeneity v_i with indicator functions V_i and bounded by the closed smooth surfaces $\Gamma_i := \partial v_i$ $(i=1,2,\ldots)$ defined by the relations $\Gamma_i(\mathbf{x})=0$ $(\mathbf{x}\in\Gamma_i)$, $\Gamma_i(\mathbf{x})>0$ $(\mathbf{x}\in v_i)$, and $\Gamma_i(\mathbf{x})<0$ $(\mathbf{x}\notin v_i)$. It is assumed that the heterogeneities can be grouped into components (phases) $v^{(q)}$ $(q=1,2,\ldots,N)$ with identical mechanical and geometrical properties (such as the shape, size, orientation, and microstructure of heterogeneities).

It is assumed that the representative macrodomain w contains a statistically large number of realizations α (providing validity of the standard probability technique) of heterogeneities $v_i\in v^{(k)}$ of the constituent $v^{(k)}$ $(i=1,2,\ldots;\ k=1,2,\ldots,N)$. A random event α belongs to a sample space $\mathcal{A}$, over which a probability density $p(\mathbf{x},\alpha)$ is defined (see, e.g., Willis, 1981). For any given α, any random function $\mathbf{g}(\mathbf{x},\alpha)$ (e.g., $\mathbf{g}=V, V^{(k)}$) is defined explicitly as one particular member, with label α, of an ensemble realization. Then, the mean, or ensemble average is defined by the angle brackets enclosing the quantity $\mathbf{g}$

$$\langle\mathbf{g}\rangle(\mathbf{x})=\int_{\mathcal{A}}\mathbf{g}(\mathbf{x},\alpha)p(\mathbf{x},\alpha)d\alpha. \tag{2.1}$$

No confusion will arise below in notation of the random quantity $\mathbf{g}(\mathbf{x},\alpha)$ if the label α is removed. One treats two material length scales (see, e.g, Torquato, 2002): the macroscopic scale L, characterizing the extent of w, and the microscopic scale a, related with the heterogeneities v_i. Moreover, one supposes that applied field varies on a characteristic length scale Λ. The limit of our interests for both the material scales and field one is

$$L\gg\Lambda\geqslant a. \tag{2.2}$$

All the random quantities under discussion are described by statistically inhomogeneous random fields. In parallel with the random indicator function $V(\mathbf{x},\alpha)$ we use the random field $V^\delta_k(\mathbf{x}-\mathbf{x}_j,\alpha)$ of delta functions placed in the inclusion centers $\mathbf{x}_k$. For the alternative description of the random structure of a composite material let us introduce a conditional probability density $\varphi(v_i,\mathbf{x}_i\mid v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)$, which is a probability densityfor finding a heterogeneity of type i with the center $\mathbf{x}_i$ in the domain v_i, given that the fixed heterogeneities $v_1,\ldots,v_n$ are centered at $\mathbf{x}_1,\ldots,\mathbf{x}_n$ (see, e.g., Willis, 1978). The configuration $(v_i,\mathbf{x}_i)$ is completely described by a detailed marked density function $\varphi(v_i,\mathbf{x}_i\mid v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)$ of the centers of an inclusion with mark v_i (which can contain information about the inclusions such as the shape, size, orientation, and material properties) being placed at $\mathbf{x}_i$ (see for details Section 5.3.1 in Buryachenko, 2007a). The notation $\varphi(v_i,\mathbf{x}_i\mid;v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)$ denotes the case $\mathbf{x}_i\neq\mathbf{x}_1,\ldots,\mathbf{x}_n$. In the case of statistically inhomogeneous media with homogeneous matrix (for so-called *Functionally Graded Materials*, FGM, see, e.g., Markworth et al., 1995; Mortensen and Suresh, 1995) the conditional probability density is not invariant with respect to translation

$$\varphi(v_i,\mathbf{x}_i\mid v_1,\mathbf{x}_1,\ldots,v_n,\mathbf{x}_n)\neq\varphi(v_i,\mathbf{x}_i+\mathbf{x}\mid v_1,\mathbf{x}_1+\mathbf{x},\ldots,v_n,\mathbf{x}_n+\mathbf{x}), \tag{2.3}$$