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Semi-analytical solution for mode I penny-shaped crack in a soft inhomogeneous layer

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ABSTRACT

An axisymmetric elastostatics problem for a penny-shaped crack placed in the middle of a inhomogeneous (FGM) elastic layer is considered. It is assumed that the elastic modulus of the layer varies through the thickness symmetrically with respect to the crack plane. Several specific distributions of the moduli variations have been analysed. We report a semi-analytical approximate solution for the determination of the stress intensity factor for the distributions considered. The obtained solution is accurate enough and can be applied in engineering applications for the analysis of crack propagation in FGM and hydrofracture growth in elastic reservoirs.

1. Introduction

The work is aimed at investigation of the fracture characteristics of elastic materials with inhomogeneous layers weakened by cracks. The classical problem for the disk-like crack in homogeneous isotropic elastic medium was studied (Sneddon, 1946) by the method of dual integral equations and (Sack, 1946) with the use of spherical harmonic functions. For two-dissimilar media the problem with an interface crack has been studied in a number of studies, among them in Arin and Erdogan (1971), Erdogan (1965), Erdogan and Arin (1972), Kassir and Bregman (1972), Lowengrub and Sneddon (1972), and Willis (1972).

The development of advanced materials, such as functionally graded materials, FGM, necessitates investigation of fracture propagation in media with non-uniform elastic properties. Particular formulations for FGM layers with continuous variations of elastic properties have been considered by Selvadurai (2000), for the case when the shear moduli of the bonded half-spaces vary in accordance with the exponential law $G(z) = G_1 + G_2 e^{\pm z\varsigma}$ (where the z-axis is perpendicular to the interface and the parameter ς characterises the rate of exponent decay/grow). The disk-like crack at a bonded plane (the interface between two half-spaces) with localised elastic inhomogeneity has been considered and the mode I stress intensity factors for different shear moduli distributions were calculated for the case of uniform remote tension applied perpendicular to the crack plane. The method used has been based on the Hankel transform followed by numerical solving a system of the Fredholm equation of the second kind.

It should be noted that the exponential form for elastic moduli is convenient for mathematical manipulations, however other forms present certain interest as for FGM as in other applications, for instance (Mendelsohn, 1984) for investigations of hydrofracture development (Savitski and Detournay, 2002) in inhomogeneous reservoirs surrounded by the rock layers with different (but constant) elastic properties. Thus, geological observations of Bazhenov shale formation structures (e.g. Strahov, 1970) and lab tests of velocity anisotropy of different shale formations (e.g., Vernik and Liu, 1997) demonstrate diversity in elastic moduli through the layer thickness. For the plane case such formulations are found, e.g., in Erdogan and Gupta (1971a,b) and more general in Delale and Erdogan (1988); for penny-shaped cracks in dissimilar layer one can mention Arin and Erdogan (1971). This study partly employs the above-mentioned formulations but assumes that the layer is spatially inhomogeneous through its thickness and deals with a mode I penny-shaped crack.

The lack of general analytical solutions for the problems involving cracks in functionally graded materials is emphasised by Eischen (1987). It should be noted that the methods of contact

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mechanics for FGM can also be applied for the crack problems. For instance, one can use the piecewise linear approximation of elastic moduli for constructing the kernel transform as suggested (Ke and Wang, 2007; Liu et al., 2008) for the contact problems for half-space and half-plane with arbitrary variations of elastic properties. We will further expand the techniques developed in a number of previous studies (Aizikovich and Alexandrov, 1984; Aizikovich, 1995; Aizikovich et al., 2002; Aizikovich et al., 2011; Vasiliev et al., 2012) for contact problems for the case of a penny-shaped crack located within a functionally graded layer. We construct a semi-analytical solution that depends on a single dimensionless parameter characterising the ratio of the crack radius to the layer thickness and examine the accuracy of such approximate solution. The analysis is conducted for a soft layer, although no restrictions is imposed for the case when the layer is stiffer that the surrounded media.

2. Formulation of the problem

Let us consider a mode I disk-like crack of certain radius R in isotropic inhomogeneous space. The crack lies in the plane $z = 0$ of the cylindrical coordinate system (r, ϕ, z) and its centre is located at the origin.

It is assumed that the elastic modulus of the space is an even function of z of the following form

$$E(z) = E_\infty \begin{cases} f(z), & 0 \leqslant |z| \leqslant H \\ 1, & |z| > H \end{cases} \tag{1.1}$$

where $f(z)$ is an arbitrary function (continuous or piecewise continuous), H is the thickness of the inhomogeneous layer and E_∞ is the modulus of the material outside the layer, see Fig. 1.

Additionally the following conditions are satisfied to provide positiveness of Young's modulus

$$\min_{z\in(0;\infty)} \Delta(z) \geqslant c_1 > 0, \quad \max_{z\in(0;\infty)} \Delta(z) \leqslant c_2 < \infty, \quad \lim_{z\to\infty} \Delta(z) = const$$
$$\Delta(z) = 2\mathrm{M}(z)(\Lambda(z) + \mathrm{M}(z))(\Lambda(z) + 2\mathrm{M}(z))^{-1} = G(z)(1 - \nu(z))^{-1} = \tfrac{1}{2}E(z)\left(1 - \nu(z)^2\right)^{-1} \tag{1.2}$$

Here $G(z)$ is the shear modulus, $\Lambda(z)$ and $\mathrm{M}(z)$ are the Lame coefficients, $\nu(z)$ is Poisson's ratio of the inhomogeneous spaces, c_1, c_2 are certain constants.

Let us further analyse the case when the crack surfaces are loaded by normal pressure $p(r) > 0$. Given symmetry of the elastic properties with respect to the z-axis and the loading conditions one can suggest that the stress/strain/displacement fields are independent of the angular coordinate φ, which leads to the following 2D axisymmetric boundary value problem for the half-space

$$\begin{aligned} &\tau_{rz}(r,0) = 0, \quad 0 < r < \infty \\ &\sigma_z(r,0) = -p(r), \quad 0 \leqslant r \leqslant R; \; w(r,0) = 0, \; r > R \end{aligned} \tag{1.3}$$

where $\sigma_z(r,z)$, $\tau_{rz}(r,z)$ are the normal and shear components of the stress tensor respectively and $w(r,z)$ is the normal component of displacements. It is also assumed that the displacements and the stresses are continuous across the planes $|z| = H$ and vanish at infinity.

It has been shown (Aizikovich and Alexandrov, 1984) that under conditions (1.2) the following relationship between the normal stresses and the normal displacements on the surface of the half-space ($z = 0$) is satisfied

$$\begin{aligned} &w(r,0) = \Delta^{-1}(0)\int_0^R q(\rho)\rho d\rho \int_0^\infty L(\gamma)J_0(\gamma\rho)J_0(\gamma r)d\gamma \\ &q(r) = \sigma_z(r,0) = \int_0^\infty Q(\alpha)J_0(\alpha r)\alpha d\alpha, Q(\alpha) = \int_0^R q(\rho)J_0(\alpha\rho)\rho d\rho \end{aligned} \tag{1.4}$$

Here $J_0(r)$ is the Bessel function and the function Δ is defined in (1.2), γ is the dimensional parameter of the Hankel transform.

The function $L(\gamma)$ is found numerically by the method of modulating functions, detail in Babeshko et al. (1987). It has the following asymptotics as shown by Aizikovich and Alexandrov (1984) (provided that the conditions specified by Eqs. (1.1) and (1.2) are valid)

$$L(\gamma) = A + B|\gamma| + \bar{\bar{o}}(\gamma^2), \quad \gamma \to 0 \tag{1.5}$$

$$L(\gamma) = 1 + D|\gamma|^{-1} + \bar{\bar{o}}(\gamma^{-2}), \quad \gamma \to \infty \tag{1.6}$$

where $A = \Delta(0)\Delta^{-1}(|H|)$ and B, D are constants. It should be noted that for a multilayer media this function possesses the following properties (Aizikovich and Alexandrov, 1982)

$$L(\alpha) = A + o(\alpha), A = D_1^{-1}D_2^{-1}\cdots D_{n-1}^{-1}, \quad \alpha \to 0 \tag{1.7}$$

$$L(\alpha) = 1 + B\left(\alpha^2 h_1^2 + \alpha h_1\right)Me^{-2\alpha h_1} + o(e^{-2\alpha h_1}), \quad \alpha \to \infty \tag{1.8}$$

Here

$$M = \frac{4(\tilde{\Delta}_1 + k_2)^2 - 1}{(D_1+1)^2 - (k_1 D_1 - k_2)^2}, \quad n \geqslant 2; \quad \tilde{\Delta}_1 = \frac{D_1}{2(1-\nu_1)};$$
$$k_j = \frac{1-\nu_j}{2(1-\nu_j)}, \quad D_k = \frac{E_{k+1}(1-\nu_k^2)}{E_k(1-\nu_{k+1}^2)}$$

and h_1 is the thickness of the upper layer, E_i and the Young moduli and the Poisson's coefficients of the j^{th} layer respectively.

It is evident that the second terms in (1.6) and (1.8) are different at $\alpha \to \infty$, which emphasise the difference in solutions of the integral equations for FGM and layered media. The properties (1.5) and (1.7) mean that the value $L(0)$ does not depend on the variation of the Lame coefficients but rather determined by their values at $z = 0$ and $|z| = H$.

Using the approach (Ishlinsky, 1986) and taking into account (1.3) one can present (1.4) in the form

$$\int_0^R \delta(\rho)\rho d\rho \int_0^\infty \frac{\gamma^2}{L(\gamma)}J_0(\gamma\rho)J_0(\gamma r)d\gamma = -\Delta^{-1}(0)p(r), \quad 0 \leqslant r \leqslant R \tag{1.9}$$

where $\delta(r) = -w(r,0)$ is the function that describe the shape of the crack (crack opening displacements, COD). This function should satisfy the following condition

$$\delta(R) = 0 \tag{1.10}$$

By taking into account the above relationships one can reduce the problem to the following dual integral equations for an auxiliary function $\Delta_1(\beta)$

$$\begin{cases} \int_0^\infty \frac{\Delta_1(\beta)}{L(\beta\lambda)}J_1(\beta r)d\beta = \Delta^{-1}(0)p^*, & 0 \leqslant r \leqslant 1 \\ \int_0^\infty \Delta_1(\beta)\beta J_1(\beta r)d\beta = 0, & r > 1 \end{cases} \tag{1.11}$$

Here the new unknown function $\Delta_1(\beta)$ is linked with the unknown crack opening displacements by the following relationship $\delta(r) = \int_0^\infty \Delta_1(\alpha)J_0(\alpha r)d\alpha$, and $J_1(r)$ is the Bessel function of the first order. In the right hand side of (1.11) we introduce a dimensionless load p^* as detailed in Appendix A. Further the asterisk at the notation for the dimensionless loads will be removed for compactness.

It is convenient to denote the reciprocal of $L(u)$ as $F(u)$ in the kernel of (1.8) and bear in mind the asymptotic behaviour of $F(u)$ yielding from (1.5) and (1.6)

$$\begin{aligned} &F(\gamma) = A^{-1} - BA^{-2}|\gamma| + \bar{\bar{o}}(\gamma^2), \quad \gamma \to 0 \\ &F(\gamma) = 1 - D|\gamma|^{-1} + \bar{\bar{o}}(\gamma^{-2}), \quad \gamma \to \infty \end{aligned} \tag{1.12}$$