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Peridynamic beams: A non-ordinary, state-based model

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ABSTRACT

This paper develops a new peridynamic state based model to represent the bending of an Euler–Bernoulli beam. This model is non-ordinary and derived from the concept of a rotational spring between bonds. While multiple peridynamic material models capture the behavior of solid materials, this is the first 1D state based peridynamic model to resist bending. For sufficiently homogeneous and differentiable displacements, the model is shown to be equivalent to Eringen's nonlocal elasticity. As the peridynamic horizon approaches 0, it reduces to the classical Euler–Bernoulli beam equations. Simple test cases demonstrate the model's performance.

1. Introduction

A goal of many mechanical engineering analyses is the prediction and description of material failure. When processes such as fracture are modeled, the partial-differential equations of classical mechanics are ill-defined at the resulting discontinuities in displacement. A peridynamic formulation of continuum mechanics casts material behavior in terms of integral functions of displacement (as opposed to gradients of displacement), so that discontinuities can evolve naturally and require no special treatment. Various peridynamic material models capture the deformation behavior of 3-dimensional solid objects (Silling et al., 2007; Silling and Askari, 2005; Gerstle et al., 2007), but would be very expensive to implement for a thin plate or beam, as the thru-thickness discretization requirement to properly capture resistance to bending would be prohibitively expensive in a computational setting for a long, slender structural object. Other peridynamic models capture tension and compression in 1D bars (Silling et al., 2003) and 2D membranes (Silling and Bobaru, 2005), but these features do not resist transverse displacement. A recent paper by Taylor and Steigmann (2013) reduces a bond based 3D plate to two dimensions with an integral through the plate's thickness. This creates a model that can represent thin structures and includes a bending term, but is used to simulate tension loading. The model is limited to the 3D bond-based Poisson ratio $\nu = \frac{1}{4}$, though the same technique could be applied to a state-based model at the expense of complexity.

This paper presents a peridynamic equivalent to an Euler–Bernoulli beam, along with a methodology for representing non-uniform cross-sections, plastic behavior, and failure. Unlike many continuum beam theories that derive new equations of motion (such as fourth order PDE's) from the 3D elastic constitutive model, the new model is not derived from prior ordinary peridynamic models based on bond extension, but is a material model that directly resists bending deformation while maintaining the same conservation of momentum equation as the 3D model. In addition to directly modeling a beam in bending, the simple beam case lays the theoretical framework for more complex peridynamic beam, plate, and shell bending models. Because many analyses of interest are partly or wholly comprised of these types of features, their development is an important addition to the capabilities of peridynamic analysis. The remainder of this introduction reviews other nonlocal work and provides a brief introduction to peridynamics, including state based models. Section 2 presents the state based beam model and demonstrates equivalence to classical Euler–Bernoulli beam theory in the limit of shrinking nonlocality. Section 3 demonstrates the beam model with simple numerical examples. Section B demonstrates the model's relationship to Eringen's nonlocal elasticity for small peridynamic horizons.

1.1. Nonlocal beam models

Nonlocal elasticity generally allows for forces at a point that are dependent on the material configuration of an entire body, rather than the configuration at that point (Eringen and Edelen, 1972). While long-range forces are obvious at the molecular model, material at larger scales is conventionally modeled as though internal forces are local or contact forces (Kröner, 1967). The result of such

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approximation is accurate for deformations that are homogeneous, but introduces some inaccuracy for inhomogeneous deformations like the propagation of waves with short wavelengths. One way to distinguish between homogeneous and inhomogeneous deformations is to incorporate higher-order gradients of deformation. While stress in classical elasticity is a function of the (first) gradient of deformation, Eringen's formulation of a nonlocal modulus in Eringen (1983) approximates a weighted sum of the first and second order gradients. This introduces a length scale to the model and has the effect of smearing out local deformation inhomogeneities over the surrounding material, while maintaining the conventional result for homogeneous deformations.

Previous work in the nonlocal mechanics of beams is motivated by the observed stiffening of nanoscale cantilevers. Challamel and Wang demonstrate in Challamel and Wang (2008) that Eringen nonlocal elasticity cannot reproduce the scale stiffening, but that stiffening does result from other gradient-elastic models and models incorporating nonlocal curvature. Because all of these models incorporate higher-order gradients of deformation, they impose stronger continuity requirements than classical elasticity, and are unsuitable for discontinuous displacements. Because the gradients are evaluated locally, gradient models are called *weakly nonlocal*. Recent work by Paola et al. (2014) develops a displacement-based beam in which relative axial displacement, shear displacement, and rotation of non-adjacent beam segments are resisted by three kinds of nonlocal spring, whose stiffnesses can be tuned to the expected material behavior. With the appropriate nonlocal stiffnesses, their model reproduces the nanoscale cantilever stiffening effect.

1.2. Peridynamics

The term *peridynamic* alludes to the fact that the force at a point is affected by nearby material configuration and was coined by Silling to describe the new formulation of continuum mechanics he developed in Silling (2000). In contrast to gradient models, the peridynamic model is *strongly nonlocal* and casts material behavior at a point as the *integral equation*

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}) = \int_{\Omega} \mathbf{f}(\mathbf{x},\mathbf{q})dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})$$

rather than the classical *partial-differential equation*. Instead of the divergence of stress, we have the integral of a "force" functional $\mathbf{f}$ of the position vectors $\mathbf{x}$ and $\mathbf{q}$ of a point within the body domain Ω. This force functional may depend on $\mathbf{x}, \mathbf{q}$, their deformed positions, the original and deformed positions of other points in Ω, history, etc.

Constitutive modeling of a wide variety of materials is accomplished by choosing the appropriate form for the force function. While the simplest force functions recreate a one-parameter linear elastic solid material (Silling, 2000), other force functions can be used to model nonlinear elasticity, plasticity, damage, and other behaviors (Silling and Bobaru, 2005).

To describe force functionals that incorporate the behavior of a totality of points in the nearby material (not just $\mathbf{x}$ and $\mathbf{q}$), we must introduce the concept of a peridynamic state.

Introduced by Silling et al. (2007), states are functions of the behavior of the continuum points surrounding each location. The most common states are scalar-states and vector-states which are scalar and vector valued, respectively. Unlike a second order tensor, which can only map vectors linearly to other vectors, vector-states can produce nonlinear or even discontinuous mappings. Important properties of states are magnitude and direction, while important operations include the addition and decomposition of states, inner and tensor products, and the Fréchet derivative of a function with respect to a state (Silling et al., 2007).

Conservation of linear momentum in the *state-based* peridynamic formulation results in the equation of motion,

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}) = \int_{\Omega} (\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle \mathbf{x}-\mathbf{q}\rangle)dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}),$$

in which $\underline{\mathbf{T}}[\,]\langle\,\rangle$ is a *force vector-state* that maps the vector in angle brackets, $\langle\rangle$, originating at the point in square brackets, [], to a force vector acting on that point. The deformed image of the vector $(\mathbf{q}-\mathbf{x})$ is defined as the *deformation vector-state*, usually denoted $\underline{\mathbf{Y}}$ and formulated as shown in Eq. (1) for a displacement field $\mathbf{u}$.

$$\underline{\mathbf{Y}}[\mathbf{x}]\langle \mathbf{q}-\mathbf{x}\rangle = (\mathbf{q}-\mathbf{x}) + (\mathbf{u}(\mathbf{q}) - \mathbf{u}(\mathbf{x})) \tag{1}$$

Just as stress and strain are work conjugate, so too are the force and deformation vector states for hyperelastic materials. If the force state $\underline{\mathbf{T}}$ is always in the same direction as the deformation state $\underline{\mathbf{Y}}$, then the force exerted by a "bond" (i.e. the vector $\mathbf{q}-\mathbf{x}$ between points is in the same direction as the deformed bond, and the model is called *ordinary*. Models in which the bond-force interactions are not in the same direction as the deformed bond are called *non-ordinary*. Silling et al. demonstrate the possibility of such models in Silling and Lehoucq (2010), but very little work has touched on their use. Foster et al. (2010) and Warren et al. (2009) show that some correspondence models, which approximate the deformation gradient and use it to calculate bond forces, result in non-ordinary state-based constitutive models for finite deformations.

2. A non-ordinary beam model

Consider the material model illustrated in Fig. 1 in which every bond-vector originating from a point is connected by a rotational spring to its opposite originating from that same point. If we call the deformed angle between these bonds θ, and choose the potential energy of that spring to be $w(\xi) = \omega(\xi)\alpha[1+\cos(\theta)]$ for the bond pair ξ and $-\xi$, we can recover the non-ordinary force state proposed by Silling et al. (2007) by taking the Fréchet derivative. For the derivation and a description of the Fréchet derivative see Appendix A.

$$\underline{\mathbf{T}}\langle\xi\rangle = \nabla w(\underline{\mathbf{Y}}\langle\xi\rangle) = \omega(\xi)\frac{-\alpha}{|\underline{\mathbf{Y}}\langle\xi\rangle|}\frac{\underline{\mathbf{Y}}\langle\xi\rangle}{|\underline{\mathbf{Y}}\langle\xi\rangle|}\times\left[\frac{\underline{\mathbf{Y}}\langle\xi\rangle}{|\underline{\mathbf{Y}}\langle\xi\rangle|}\times\frac{\underline{\mathbf{Y}}\langle-\xi\rangle}{|\underline{\mathbf{Y}}\langle-\xi\rangle|}\right] \tag{2}$$

Though it looks complex, Eq. (2) indicates a bond force perpendicular to the deformed bond and in the plane containing both the deformed bond and its partner as illustrated in Fig. 2. The force magnitude is proportional to the sine of the angle between the bonds divided by the length of the deformed bond. This response is consistent with the idea of a rotational spring between bonds as long as the change in angle is small. Because the potential energy and force states are functions of *pairs* of peridynamic bonds, we will call this formulation a *bond-pair model*. Other choices for the bond-pair potential function, such as $w = (\pi-\theta)^2$, are also possible, but result in more mathematically complex analysis.

2.1. Energy equivalence

To determine an appropriate choice of α, we desire our peridynamic model to have an equivalent strain energy density to a classical Euler–Bernoulli beam in the *local limit*, i.e. when the nonlocal length scale vanishes. We will begin with the assumptions from

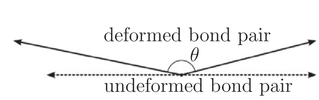

Fig. 1. Illustration of a bond pair model that resists angular deformation.