International Journal of Solids and Structures 51 (2014) 2921–2930

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Green's functions for multi-phase isotropic laminated plates

Xu Wang [a,*], Kun Zhou [b,*]

[a] School of Mechanical and Power Engineering, East China University of Science and Technology, 130 Meilong Road, Shanghai 200237, China
[b] School of Mechanical and Aerospace Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore

ARTICLE INFO

Article history:
Received 8 November 2013
Received in revised form 23 April 2014
Available online 6 May 2014

Keywords:
Green's function
Kirchhoff isotropic laminated plate
Bimaterial
Circular elastic inclusion
Composite space

ABSTRACT

This work develops a series of Green's functions for multi-phase Kirchhoff isotropic laminated plates. First, we derive the Green's functions for a composite laminated plate composed of two bonded dissimilar isotropic laminated semi-infinite plates. Second, the obtained results for bimaterials are judiciously applied to obtain the Green's function solution for a circular elastic inclusion embedded in an infinite isotropic laminated plate. Third, Green's functions for a composite space composed of an arbitrary number of wedges of different isotropic laminated plates are derived. Finally, we derive Green's functions for a laminated plate with an elliptical and a parabolic boundary, respectively.

1. Introduction

The Green's function solutions are essential in the development of the theory of elasticity. They can be employed to study Eshelby's inclusion problems (Eshelby, 1957; Seo and Mura, 1979; Chiu, 1980; Gharpuray et al., 1991; Wu and Du, 1995; Rodin, 1996; Nozaki and Taya, 1997; Beom and Earmme, 1999; Beom and Kim, 1999; Xu and Wang, 2005; Zou et al., 2010) and to investigate crack problems (Erdogan et al., 1974; Erdogan and Gupta, 1975; Patton and Santare, 1990; Anlas and Santare, 1993; Wang and Hasebe, 2000), anticrack problems (Dundurs and Markenscoff, 1989; Markenscoff et al., 1994) and debonded anticrack problems (Markenscoff and Ni, 1996).

The Green's function solutions for isotropic laminated plates have rarely been addressed except the early result by Beom and Earmme (1998). Very recently, the present authors (Wang and Zhou, 2014a) derived Green's function solutions for an infinite and a semi-infinite Kirchhoff isotropic laminated plate subjected to concentrated forces, concentrated moments, discontinuous displacements and slopes by using an elegant complex variable formulation for isotropic laminated plates (Beom and Earmme, 1998; Wang and Zhou, 2014b).

This study aims to derive Green's function solutions for multi-phase Kirchhoff isotropic laminated plates. More specifically, we will derive: (i) Green's functions for a bimaterial composed of two bonded dissimilar isotropic laminated semi-infinite plates; (ii) Green's functions for a circular elastic inclusion embedded in an infinite isotropic laminated plate; (iii) Green's functions for a composite space composed of an arbitrary number of wedges of different isotropic laminated plates; (iv) Green's functions for an isotropic laminated plate containing an elliptical hole or elliptical rigid inclusion; (v) Green's functions for an isotropic laminated plate with a parabolic boundary.

The obtained Green's functions can be used to formulate boundary integral method to solve general laminar plate problems. Moreover, the Green's functions can be used to derive the Eshebly tensors for homogenization of heterogeneous laminar plates, which have been done only for classical thin plates and Mindlin plates (see for example, Li, 2000a,b). It is expected that the obtained analytical results can also find potential applications in modern heterogeneous membranes (Mohammadi et al., 2014). In fact, Mohammadi et al. (2014) only addressed a simpler case in which the out-of-plane deformation is decoupled from the in-plane behavior.

2. Complex variable formulation

Consider a plate of uniform thickness h composed of isotropic linear elastic material that can be inhomogeneous and laminated in the thickness direction. The main plane of the plate before deformation is located at $x_3 = 0$ in a Cartesian coordinate system $\{x_i\}(i = 1, 2, 3)$, which is chosen such that the two in-plane displacements u_α and the out-of-plane deflection w on the main plane are decoupled in the equilibrium equations (Beom and Earmme, 1998). This makes it possible to uniquely determine the distance

* Corresponding authors. Tel.: +65 67905499; fax: +65 67924062 (K. Zhou), tel: +86 21 64251805; fax: +86 21 64251805 (X. Wang).
E-mail addresses: xuwang@ecust.edu.cn (X. Wang), kzhou@ntu.edu.sg (K. Zhou).

http://dx.doi.org/10.1016/j.ijsolstr.2014.04.022
0020-7683/

h_0 between the main plane and the lower surface of the plate (Beom and Earmme, 1998). It is noted that the Greek indices range from 1 to 2 hereafter.

The membrane stress resultants $N_{\alpha\beta} = Q\sigma_{\alpha\beta}$, bending moments $M_{\alpha\beta} = Qx_3\sigma_{\alpha\beta}$ with $\sigma_{\alpha\beta}$ being the in-plane stresses and $Q(\cdots) = \int_{-h_0}^{h-h_0}(\cdots)\mathrm{d}x_3$, transverse shearing forces $R_\beta = M_{\alpha\beta,\alpha}$, in-plane displacements, out-of-plane deflection and slopes $\vartheta_\alpha = -w_{,\alpha}$ on the main plane of the plate, and the four stress functions φ_α and η_α can be expressed by four complex potentials $\phi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ with $z = x_1 + ix_2 = re^{i\theta}$ as follows (Beom and Earmme, 1998; Cheng and Reddy, 2002; Wang and Zhou, 2014b):

$$
\begin{aligned}
&N_{11}+N_{22}=4\mathrm{Re}\{\phi'(z)+B\Phi'(z)\},\\
&N_{22}-N_{11}+2iN_{12}=2\left[\bar{z}\phi''(z)+\psi'(z)+B\bar{z}\Phi''(z)+B\Psi'(z)\right],\\
&M_{11}+M_{22}=4D(1+\nu^D)\mathrm{Re}\{\Phi'(z)\}+\frac{B(\kappa^A-1)}{\mu}\mathrm{Re}\{\phi'(z)\},\\
&M_{22}-M_{11}+2iM_{12}=-2D(1-\nu^D)\left[\bar{z}\Phi''(z)+\Psi'(z)\right]-\frac{B}{\mu}[\bar{z}\phi''(z)+\psi'(z)],\\
&R_1-iR_2=4D\Phi''(z)+\frac{B(\kappa^A+1)}{2\mu}\phi''(z),
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&2\mu(u_1+iu_2)=\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)},\\
&\vartheta_1+i\vartheta_2=\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}, w=-\mathrm{Re}\{\bar{z}\Phi(z)+\chi(z)\},\\
&\phi_1+i\phi_2=i\left[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}\right]+iB\left[\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}\right],\\
&\eta_1+i\eta_2=iD(1-\nu^D)\left[\kappa^D\Phi(z)-z\overline{\Phi'(z)}-\overline{\Psi(z)}\right]+i\frac{B}{2\mu}\left[\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}\right],
\end{aligned}
\tag{2}
$$

where $\Psi(z) = \chi'(z)$, and

$$
\begin{aligned}
&\mu=\frac{1}{2}(A_{11}-A_{12}),\quad B=B_{12},\quad D=D_{11},\quad \nu^A=\frac{A_{12}}{A_{11}},\\
&\nu^D=\frac{D_{12}}{D_{11}},\quad \kappa^A=\frac{3A_{11}-A_{12}}{A_{11}+A_{12}}=\frac{3-\nu^A}{1+\nu^A},\\
&\kappa^D=\frac{3D_{11}+D_{12}}{D_{11}-D_{12}}=\frac{3+\nu^D}{1-\nu^D}.
\end{aligned}
\tag{3}
$$

The five elastic constants $A_{11}, A_{12}, B_{12}, D_{11}$ and D_{12} refer to the work by Beom and Earmme (1998). The membrane stress resultants, bending moments, transverse shearing forces, and modified Kirchhoff transverse shearing forces $V_1 = R_1 + M_{12,2}$ and $V_2 = R_2 + M_{21,1}$ can be expressed by the four stress functions φ_α and η_α (Cheng and Reddy, 2002):

$$
\begin{aligned}
&N_{\alpha\beta}=-\epsilon_{\beta\omega}\varphi_{\alpha,\omega},\\
&M_{\alpha\beta}=-\epsilon_{\beta\omega}\eta_{\alpha,\omega}-\frac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega},\quad R_\alpha=-\frac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega\beta},\quad V_\alpha=-\epsilon_{\alpha\omega}\eta_{\omega,\omega\omega},
\end{aligned}
\tag{4}
$$

where $\epsilon_{\alpha\beta}$ are the components of the two-dimensional permutation tensor.

If a new coordinate system $\{\breve{x}_i\}$ $(i = 1, 2, 3)$ is established where $\breve{x}_3 = 0$ is on the mid-plane and $\breve{x}_\alpha = x_\alpha$, the in-plane displacements $\breve{u}_\alpha$ and slopes $\breve{\vartheta}_\alpha$ on the mid-plane, and the stress functions $\breve{\phi}_\alpha$ and $\breve{\eta}_\alpha$ in the new coordinate system can simply be given by Wang and Zhou (2014c)

$$
\begin{aligned}
&\breve{\vartheta}_1+i\breve{\vartheta}_2=\vartheta_1+i\vartheta_2,\quad \breve{u}_1+i\breve{u}_2=u_1+iu_2-\hat{h}(\vartheta_1+i\vartheta_2),\\
&\breve{\varphi}_1+i\breve{\varphi}_2=\varphi_1+i\varphi_2,\quad \breve{\eta}_1+i\breve{\eta}_2=\eta_1+i\eta_2+\hat{h}(\varphi_1+i\varphi_2),
\end{aligned}
\tag{5}
$$

where

$$
\hat{h}=\frac{h}{2}-h_0.
\tag{6}
$$

In the new coordinate system, the stress resultants $\breve{N}_{\alpha\beta} = \breve{Q}\,\sigma_{\alpha\beta}$ and $\breve{M}_{\alpha\beta} = \breve{Q}\,\breve{x}_3\sigma_{\alpha\beta}$ with $\breve{Q}(\cdots) = \int_{-h/2}^{h/2}(\cdots)\mathrm{d}\breve{x}_3$, transverse shearing forces $\breve{R}_\beta = \breve{M}_{\alpha\beta,\alpha}$, and modified Kirchhoff transverse shearing forces $\tilde{V}_1 = \tilde{R}_1\tilde{M}_{12,2}$ and $\tilde{V}_2 = \tilde{R}_2\tilde{M}_{21,1}$ can also be expressed in terms of the introduced stress functions $\breve{\varphi}_\alpha$ and $\tilde{\eta}_\alpha$ as

$$
\begin{aligned}
&\breve{N}_{\alpha\beta}=-\epsilon_{\beta\omega}\breve{\varphi}_{\alpha,\omega},\\
&\breve{M}_{\alpha\beta}=-\epsilon_{\beta\omega}\breve{\eta}_{\alpha,\omega}-\frac{1}{2}\epsilon_{\alpha\beta}\breve{\eta}_{\omega,\omega},\quad \breve{R}_\alpha=-\frac{1}{2}\epsilon_{\alpha\beta}\breve{\eta}_{\omega,\omega\beta},\quad \breve{V}_\alpha=-\epsilon_{\alpha\omega}\breve{\eta}_{\omega,\omega\omega}.
\end{aligned}
\tag{7}
$$

3. Green's functions for bimaterials

Consider two dissimilar isotropic laminated semi-infinite plates perfectly bonded along the real axis, as shown in Fig. 1. The mid-planes of the two semi-infinite plates are on the same plane. It is assumed that the singularities are located at $z = z_0$ $(\mathrm{Im}\{z_0\} > 0)$ in the upper semi-infinite plate. The subscripts 1 and 2 (or the superscripts (1) and (2)) are used to identify the associated quantities in the upper and lower semi-infinite plates, respectively.

The continuity conditions of displacements and stress resultants across the perfect interface $x_2 = 0$ can be more specifically expressed into

$$
\begin{aligned}
&\breve{u}_1^{(1)}+i\breve{u}_2^{(1)}=\breve{u}_1^{(2)}+i\breve{u}_2^{(2)},\quad \breve{\varphi}_1^{(1)}+i\breve{\varphi}_2^{(1)}=\breve{\varphi}_1^{(2)}+i\breve{\varphi}_2^{(2)},\\
&\breve{\vartheta}_1^{(1)}+i\breve{\vartheta}_2^{(1)}=\breve{\vartheta}_1^{(2)}+i\breve{\vartheta}_2^{(2)},\quad \breve{\eta}_1^{(1)}+i\breve{\eta}_2^{(1)}=\breve{\eta}_1^{(2)}+i\breve{\eta}_2^{(2)}.
\end{aligned}
\tag{8}
$$

In order to solve the boundary value problem containing a straight interface, it is more convenient to introduce the following new analytic functions:

$$
\Theta_k(z)=z\phi_k'(z)+\psi_k(z),\quad \Omega_k(z)=z\Phi_k'(z)+\Psi_k(z),\quad (k=1,2).
\tag{9}
$$

The principal (or singular) parts of $\phi_1(z)$, $\Phi_1(z)$, $\Theta_1(z)$ and $\Omega_1(z)$ defined in the upper half-plane are $\phi_0(z)$, $\Phi_0(z)$, $\Theta_0(z)$ and $\Omega_0(z)$, respectively; whilst $\phi_2(z)$, $\Phi_2(z)$, $\Theta_2(z)$ and $\Omega_2(z)$ defined in the lower half-plane are all regular. Explicit expressions of $\phi_0(z)$, $\Phi_0(z)$, $\Theta_0(z)$ and $\Omega_0(z)$ for different singularities have been derived by Wang and Zhou (2014a) and are listed in Appendix A.

In view of Eqs. (2), (5) and (9), the continuity conditions of displacements and stress resultants across the perfect interface $x_2 = 0$ in Eq. (8) can be expressed in terms of $\phi_k(z)$, $\Phi_k(z)$, $\Theta_k(z)$ and $\Omega_k(z)$ $(k = 1, 2)$ as

$$
\Phi_1^+(z)+\bar{\Omega}_1^-(z)=\Phi_2^-(z)+\bar{\Omega}_2^+(z),
$$

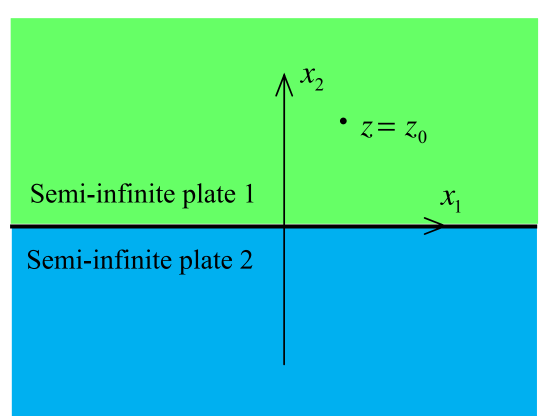

Fig. 1. A bimaterial composed of two bonded dissimilar isotropic laminated semi-infinite plates. The singularities are located in the upper semi-infinite plate 1.