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On the interpretation of the logarithmic strain tensor in an arbitrary system of representation

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ABSTRACT

Logarithmic strains are increasingly used in constitutive modelling because of their advantageous properties. In this paper we study the physical interpretation of the components of the logarithmic strain tensor in any arbitrary system of representation, which is crucial in formulating meaningful constitutive models. We use the path-independence property of total logarithmic strains to propose different fictitious paths which can be interpreted as a sum of infinitesimal engineering strain tensors. We show that the angular (engineering) distortion measure is arguably not a good measure of shear and instead we propose area distortions which are an exact interpretation of the shear terms both for engineering and for logarithmic strains. This new interpretation clearly explains the maximum obtained in some constitutive models for the simple shear load case.

1. Introduction

Traditional constitutive modelling is frequently developed for small (engineering) strains (Bathe, 1996; Kojić and Bathe, 2005). The extension of these models to large strains is not obvious. There are many fundamental issues when extending such models to large strains, as for example objectivity and energy preservation during elastic deformation processes (Eshraghi et al., 2013a,b; Holzapfel, 2000; Ogden, 1984), which do not usually deserve special attention for small strains. One important decision to be made at large strains is which stress and strain measures to employ.

In the small strain kinematically linear context the engineering infinitesimal stress and strain measures are the ones employed because distinction is not relevant among the different measures. Engineers are used to engineering strains, so they have a rather deep understanding of the physical meaning of their components. In the large strain context, unfortunately there are many choices for stress and strain measures and, of course, that choice strongly affects the constitutive equations of the model, which is usually formulated with a given strain measure in mind. Of course one strain measure may always be mapped to any other strain measure, but for example, a constitutive equation linear in one strain measure will not be so in any other measure. Hence, some fundamental conclusions obtained using one measure may not be valid using others.

The Green–Lagrange and Almansi–Euler deformation measures are often used because of two reasons: they are directly obtained from the deformation gradient and they naturally appear in the nonlinear terms of the finite element formulations. However, these deformation measures are not intuitive, even for uniaxial loading, so using them in constitutive equations may bring difficulties interpreting results or material constants of the models.

The large strain measures arguably most intuitive are the logarithmic (Hencky or "true") strain measures. As we will briefly review below, they preserve the physical meaning of the trace operator (and hence the volumetric and deviatoric strains), they are additive in uniaxial situations and they are symmetric respect to the percentage of stretching: doubling the length of an specimen gives the same amount of logarithmic strain than halving the length of the specimen, except for the change of sign. For logarithmic strains, the push-forward and pull-back operations are performed using rotations, so they also preserve the metric. Furthermore, in isotropic metals a linear hyperelastic relationship between logarithmic strains and Kirchhoff stresses has been found to be an accurate representation if the elastic strains are not too large but only moderately large (Anand, 1979, 1986). This fact added to the special structure of the exponential tensor operators on logarithmic strains facilitate enormously the formulation of elastoplastic constitutive models that are physically well grounded, accurate and efficient for finite element implementation, both for the isotropic (Eshraghi et al., 2010; Eterović and Bathe, 1990; Montáns and Bathe, 2005; Perić et al., 1992; Simó, 1992; Weber and Anand, 1990) and anisotropic cases (Caminero et al., 2011; Miehe et al., 2002; Papadopoulus and Lu, 1998). It has been shown that logarithmic strains appear naturally as a

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consequence of the combination of hypoelasticity and hyperelasticity into a single equation in the context of elastoplasticity (Xiao et al., 2007).

Logarithmic strain measures are also increasingly being used in highly nonlinear hyperelasticity to model the behavior of elastomers and living tissues. For example, recent models based on spline interpolation of experimental data are formulated using logarithmic strains, both for isotropic materials (Sussman and Bathe, 2009) and for anisotropic materials (Latorre and Montáns, 2013; Latorre and Montáns, 2014). However these models necessitate some experimental data, which must be correctly interpreted. The correct interpretation of the components of the logarithmic strain tensor in any system of representation is a key for obtaining a correct and accurate description for such models. Furthermore, as we show below, if a good understanding of the strain tensor is achieved, some useful expressions involving functions of such tensor may be obtained (Hoger, 1986; Jog, 2008).

The purpose of this paper is to make some progress in the interpretation of the components of the logarithmic strain tensor in any system of representation, paying special attention to the off-diagonal terms, and to link some conclusions with observed phenomena in the literature when these measures are being used. In particular, we are specially interested in elucidating a correct meaning and a correct measure for the shear deformation. This is of crucial importance in constitutive modelling.

The layout of the paper is as follows. First we briefly review some well-known facts about general strains with the objective of properly motivate the definition and the construction of the logarithmic strain tensor in such a way that the components of the tensor may be better understood. Then we analyze some typical shear deformation examples in order to explain the geometrical meaning of the logarithmic strain measures and to understand the limitations of these shear tests when used in constitutive modelling.

2. General strain measures

The strain measure of a uniformly stretched longitudinal rod with initial (time t_0) and current (time t) total lengths L_0 and L, respectively, may be expressed in multiple ways. It is well-known that all those usual strain measures are given by the general Seth-Hill formula (Seth, 1964)

$$E_n = \frac{1}{n}(\lambda^n - 1) \tag{1}$$

where $\lambda = \partial x(X,t)/\partial X = L/L_0$ is the current stretch ratio, n is a number that characterizes each uniaxial strain measure and $x(X,t)$ represents the motion of material points $X \in [0, L_0]$ at time t. The identity $\lambda = L/L_0$ holds due to the homogeneous deformation assumed along the rod. As it is widely known, the general formula given in Eq. (1) can be used to locally define the strains in principal directions of a three-dimensional deformation state. In that way, Eq. (1) is generalized to

$$\begin{cases} \mathbf{E}_n = \sum_{i=1}^{3} \frac{1}{n}(\lambda_i^n - 1)\mathbf{N}_i \otimes \mathbf{N}_i & \text{if } n \neq 0 \\ \mathbf{E}_0 = \sum_{i=1}^{3} \ln \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i & \text{if } n = 0 \end{cases} \tag{2}$$

where λ_i are the principal stretches and $\mathbf{N}_i$ are the principal directions of the stretch tensor $\mathbf{U}$ obtained from the right polar decomposition, or equivalently

$$\begin{cases} \mathbf{E}_n = \frac{1}{n}(\mathbf{U}^n - \mathbf{I}) & \text{if } n \neq 0 \\ \mathbf{E}_0 = \ln \mathbf{U} & \text{if } n = 0 \end{cases} \tag{3}$$

with $\mathbf{I}$ being the second-order identity tensor.

From Eqs. (2), one can easily calculate all the strain tensors $\mathbf{E}_n$, including the case $n = 0$, using the principal stretches λ_i and the eigenvectors $\mathbf{N}_i$ (previously computed). This way, since $(E_n)_i = \mathbf{N}_i \cdot \mathbf{E}_n \mathbf{N}_i = (\lambda_i^n - 1)/n$, any possible physical meaning for unidimensional strains can obviously be interpreted in the same manner along the principal stretching directions in the three-dimensional case. However, from Eqs. (2) expressed in that way, nothing can be said about the components of $\mathbf{E}_n$ when these tensors are represented in a general basis.

In order to understand the description of the cases $n \neq 0$ in a general system of representation (not only in principal directions), the general expression given in Eq. $(3)_1$ for $\mathbf{E}_n$ can be used. We will use the deformation gradient $\mathbf{U} = \partial\bar{\mathbf{x}}(\mathbf{X},t)/\partial\mathbf{X}$, where $\bar{\mathbf{x}}(\mathbf{X},t)$ represents the motion of material points $\mathbf{X}$ with the rotation $\mathbf{R}$ removed, which yields a compatible homogeneous rotationless deformation. Hence, for example, the Biot strain tensor, obtained for $n = 1$, is $\mathbf{E}_1 = \mathbf{U} - \mathbf{I} = \partial\bar{\mathbf{u}}(\mathbf{X},t)/\partial\mathbf{X}$, where it can be seen that $\mathbf{E}_1$ represents the material gradient of the displacement field $\bar{\mathbf{u}}(\mathbf{X},t) = \bar{\mathbf{x}}(\mathbf{X},t) - \mathbf{X}$. For any pair of orthogonal unit vectors $\mathbf{P}$ and $\mathbf{Q}$ in the reference configuration, see Fig. 1, we have

$$(E_1)_{PQ} = \mathbf{P} \cdot \mathbf{E}_1 \mathbf{Q} = \mathbf{P} \cdot \frac{\partial\bar{\mathbf{u}}(\mathbf{X},t)}{\partial\mathbf{X}}\mathbf{Q} = \mathbf{P} \cdot \Delta_{\mathbf{Q}} \tag{4}$$

which reveals the meaning of the components of $\mathbf{E}_1$ in a reference frame in which $\mathbf{P}$ and $\mathbf{Q}$ are basis vectors, that is, $(E_1)_{PQ}$ is the projection onto the $\mathbf{P}$ direction of the relative displacement $\Delta_{\mathbf{Q}} = \bar{\mathbf{u}}(\mathbf{X}+\mathbf{Q},t) - \bar{\mathbf{u}}(\mathbf{X},t)$ when the deformation is assumed to be homogeneous in the solid. Note that if $\mathbf{P}$ is not a principal direction of deformation, the diagonal components of $\mathbf{E}_1$ can not be understood as in the associated unidimensional case, that is, in general

$$(E_1)_{PP} \neq \lambda_P - 1 \tag{5}$$

where $\lambda_P = |\mathbf{p}|$, being $\mathbf{p} = \mathbf{UP}$ the transformed vector into the current configuration corresponding to the basis vector $\mathbf{P}$. Aside, in this case in which the rotation $\mathbf{R}$ is removed, $\mathbf{E}_1$ is equivalent to the engineering strain tensor $\boldsymbol{\varepsilon} = sym(\partial\bar{\mathbf{u}}/\partial\mathbf{X}) = \partial\bar{\mathbf{u}}/\partial\mathbf{X} = \mathbf{E}_1$. However, the well-known physical descriptions of the diagonal and off-diagonal components of $\boldsymbol{\varepsilon}$ ($\varepsilon_{PP} \approx \lambda_P - 1$ and $\varepsilon_{PQ} \approx \gamma_{PQ}/2$, being γ_{PQ} the angular distortion associated to directions $\mathbf{P}$ and $\mathbf{Q}$) can only be assigned to $\mathbf{E}_1$ if $|\mathbf{u}| \ll 1$, that is within the small strain framework.

The values $n = 2$ and $n = -2$ provide the well-known Green–Lagrange and Euler–Almansi strain tensors, respectively. If the first of them is expressed by means of Eq. (3), it results in $\mathbf{E}_2 = 1/2(\mathbf{U}^2 - \mathbf{I})$. As before, one can get a physical interpretation of the PQ-component of $\mathbf{E}_2$ when this last expression is pre- and post-multiplied by the orthogonal material basis vectors $\mathbf{P}$ and $\mathbf{Q}$. Proceeding in that way

$$(E_2)_{PQ} = \mathbf{P} \cdot \mathbf{E}_2 \mathbf{Q} = \begin{cases} \frac{1}{2}(\lambda_P^2 - 1) & \text{if } P = Q \\ \frac{1}{2}\lambda_P\lambda_Q \cos\theta_{PQ} & \text{if } P \neq Q \end{cases} \tag{6}$$

with $\lambda_P = |\mathbf{p}|$ and $\lambda_Q = |\mathbf{q}|$. In this case, unlike for $\mathbf{E}_1$ – see Eq. (3) – the diagonal terms of $\mathbf{E}_2$ correspond to the unidimensional E_2-strain measures of the fibers initially located along the reference frame axes. In a general situation, however, these fibers are not disposed

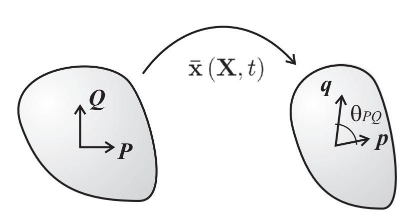

Fig. 1. Deformation of two arbitrary orthogonal directions.