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Asymptotic mechanical fields at the tip of a mode I crack in rubber-like solids

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ABSTRACT

Asymptotic analyses of the mechanical fields in front of stationary and propagating cracks facilitate the understanding of the mechanical and physical state in front of crack tips, and they enable prediction of crack growth and failure. Furthermore, efficient modelling of arbitrary crack growth by use of XFEM (extended finite element method) requires accurate knowledge of the asymptotic crack tip fields. In the present work, we perform an asymptotic analysis of the mechanical fields in the vicinity of a propagating mode I crack in rubber. Plane deformation is assumed, and the material model is based on the Langevin function, which accounts for the finite extensibility of polymer chains. The Langevin function is approximated by a polynomial, and only the term of the highest order contributes to the asymptotic solution. The crack is predicted to adopt a wedge-like shape, i.e. the crack faces will be straight lines. The angle of the wedge and the order of the stress singularity depend on the hardening of the strain energy function. The present analysis shows that in materials with a significant hardening, the inertia term in the equations of motion becomes negligible in the asymptotic analysis. Hence, there is no upper theoretical limit to the crack speed.

1. Introduction

In the vicinity of crack tips, the stress and strain fields may (theoretically) become singular, and the mechanical state in such regions may be characterised by an asymptotic solution. The nature of such singular fields reveals a great deal about the material and the mechanical state at the crack tip, and may be used to predict the risk of crack growth and failure. Furthermore, singular fields and asymptotic solutions may be used in the so-called *extended finite element method* (XFEM) (Belytschko et al., 2009; Yazid et al., 2009). The asymptotic crack tip deformation fields are then added to the standard FE shape functions, which enables the prediction of crack propagation along arbitrary paths using a relatively coarse mesh.

Analytic studies of crack tip fields go back to the 1950s. The asymptotic elastostatic crack tip fields in linearly elastic solids (Williams, 1959), as well as solids exhibiting a non-linear material behaviour (Hutchinson, 1968; Rice and Rosengren, 1968), have been considered. In addition, dynamic crack propagation in linearly elastic solids has been analysed (e.g. Sih, 1970; Clifton and Freund, 1974; Nilsson, 1974; Freund, 1990; Broberg, 1999).

With regard to rubber and soft elastic materials, Wong and Shield (1969) employed a fully non-linear theory (i.e. for both material and geometry) to analyse the deformation fields at a mode I crack tip. They adopted an incompressible neoHookean material model to analyse the deformation of a membrane under plane stress. Using a similar framework, Knowles and Sternberg (1973, 1974) studied the elastostatic mechanical fields in the vicinity of a crack tip for a homogeneous, isotropic, hyperelastic, and compressible material. Le and Stumpf (1993) performed a similar study as Knowles and Sternberg, but applied a different strain energy function for the material. Also Stephenson (1982) used a similar type of approach as Knowles and Sternberg (1973), but considered an incompressible material and introduced the hydrostatic pressure as an additional field variable. Knowles and Sternberg (1983) have also studied the deformation of a crack tip in an incompressible thin neoHookean sheet under plane stress conditions. Recently, Kroon (2011a) considered dynamic crack propagation in rubber. In this study, the same material law was used as in Knowles and Sternberg (1973, 1974), and the influence of inertia on the crack tip fields was examined.

Mixed-mode (mode I and II) cases have also been considered (Stephenson, 1982; Geubelle and Knauss, 1994), and it was concluded that the theory for large deformations excludes the possibility of a pure antisymmetric (mode II) deformation mode (Knowles, 1981; Stephenson, 1982; Geubelle and Knauss, 1994).

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The anti-plane (mode III) crack problem has also been analysed using finite strain theory (Knowles, 1977; Knowles and Sternberg, 1980, 1981; Silling, 1988a,b).

The elasto-static – and to some extent the dynamic – crack tip fields in solids undergoing finite deformations have been thoroughly examined, as indicated above. However, the non-linear nature of the problem implies that there are no unique solutions available. For instance, the resulting singular solutions depend on the choice of material law. There is some experimental evidence that crack profiles in rubber assume a parabolic shape (e.g. Gent and Marteny, 1982; Al-Quraishi and Hoo Fatt, 2007). This outcome is predicted for example in the studies by Knowles and Sternberg (1973, 1974) and Kroon (2011a). However, other experimental studies (e.g. Deegan et al., 2002; Petersan et al., 2004; Zhang et al., 2009; Chen et al., 2011) suggest that the crack profiles would be more or less straight, i.e. the crack would assume a wedge-like shape. In fact, one of these studies (Zhang et al., 2009) indicates that the shape of the crack tip may change with the crack speed and applied boundary conditions.

In the present work, we examine the mode I crack a bit further, and crack propagation under plane deformation conditions is considered. We adopt a material law that is suitable for rubber and is based on Langevin statistics for the stretching of polymer chains. However, we use the polynomial approximation of the Langevin function, and are therefore not considering the *true* Langevin function, in which there is a maximum locking stretch at which the stress/force response goes to infinity. Since the deformation of individual polymer chains is never fully affine, complete locking cannot be expected to occur in a real material. We therefore believe that from a physical point of view, the polynomial approximation of the Langevin function is more plausible than the exact function itself, since the approximation in effect adds some extra compliance at high stretches which prohibits complete locking.

In Sections 2 and 3, the eigenvalue problem is formulated, which includes the kinematics of the problem, the constitutive model, the boundary conditions applied, equations of motion, and energy relations. The numerical solution to the problem is then provided in Section 4. Finally, Section 5 provides a discussion and some concluding remarks.

2. Problem formulation

2.1. Geometry and kinematics

In the present analysis, we consider a crack that propagates through a plane structure, as illustrated in Fig. 1. A mode I crack is modelled, the crack propagates with the Lagrangian crack speed V_c, and both quasi-static and dynamic crack tip fields are considered. Three coordinate systems are introduced: a Cartesian coordinate system (X_1^0, X_2^0, X_3^0) that is fixed in space, a Cartesian coordinate system (X_1, X_2, X_3) that moves with the crack tip, and a cylindrical coordinate system (R, Θ, Z) that also moves with the crack tip. All three coordinate systems are associated with the undeformed state of the rubber material. The two moving systems have their origin at the tip of the crack, and the crack propagates along the X_1-direction. We assume that in the vicinity of the crack tip, asymptotic solutions dominate the mechanical fields and steady-state conditions prevail, such that the two Cartesian systems relate according to

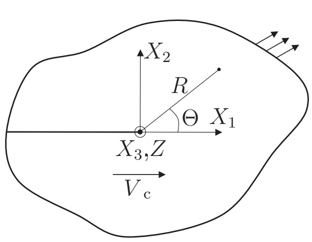

Fig. 1. Geometry and coordinates of crack problem in the reference configuration.

$$X_1^0 = X_1 + V_c \cdot t, \quad X_2^0 = X_2, \quad X_3^0 = X_3, \tag{1}$$

where t denotes time. Differentiation of Eq. $(1)_1$ yields the relation $\mathrm{d}t = -\mathrm{d}X_1/V_c$, which enables us to rewrite time derivatives of field variables according to

$$\frac{\mathrm{d}(\bullet)}{\mathrm{d}t} = \frac{\partial(\bullet)}{\partial X_1}\frac{\mathrm{d}X_1}{\mathrm{d}t} = -V_c\frac{\partial(\bullet)}{\partial X_1}. \tag{2}$$

The coordinates of the moving coordinate systems relate according to

$$X_1 = R \cdot \cos\Theta, \quad X_2 = R \cdot \sin\Theta, \quad X_3 = Z \tag{3}$$

and

$$R = \sqrt{X_1^2 + X_2^2}, \quad \Theta = \arctan\left(\frac{X_2}{X_1}\right), \tag{4}$$

where $X_1, X_2, X_3, Z \in (-\infty, \infty), R \in [0, \infty)$, and $\Theta \in [-\pi, \pi]$.

Henceforth, we only consider the coordinate systems that move with the crack tip. The position vector in the undeformed configuration is denoted by $\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ is a set of orthogonal unit vectors associated with the three coordinates X_1, X_2, and X_3, respectively. The position vector in the deformed configuration is denoted by $\mathbf{x} = \mathbf{X} + \mathbf{u}$, where $\mathbf{u}$ is the displacement vector. The deformation gradient is defined as $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, and the right Cauchy–Green deformation tensor is $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$.

For the asymptotic deformation field at the crack tip, the following ansatz is proposed:

$$\begin{aligned} x_1 &= R^\alpha \cdot f_1(\Theta) + x_0, \\ x_2 &= R^\beta \cdot f_2(\Theta), \\ x_3 &= x_3(Z). \end{aligned} \tag{5}$$

Stress fields perpendicular to the crack are expected to be singular $(0 < \beta < 1)$. Stress fields along the crack extension may also be singular but possibly with a higher exponent $(\alpha \geqslant \beta)$, associated with a weaker singularity.

The components of the deformation gradient are computed according to

$$F_{ij} = \frac{\partial x_i}{\partial R}\frac{\partial R}{\partial X_j} + \frac{\partial x_i}{\partial\Theta}\frac{\partial\Theta}{\partial X_j} + \frac{\partial x_i}{\partial Z}\frac{\partial Z}{\partial X_j}, \tag{6}$$

where the partial derivatives are

$$\begin{array}{lll} \frac{\partial R}{\partial X_1} = \cos\Theta, & \frac{\partial R}{\partial X_2} = \sin\Theta, & \frac{\partial R}{\partial X_3} = 0, \\ \frac{\partial\Theta}{\partial X_1} = -\frac{\sin\Theta}{R}, & \frac{\partial\Theta}{\partial X_2} = \frac{\cos\Theta}{R}, & \frac{\partial\Theta}{\partial X_3} = 0, \\ \frac{\partial Z}{\partial X_1} = 0, & \frac{\partial Z}{\partial X_2} = 0, & \frac{\partial Z}{\partial X_3} = 1. \end{array} \tag{7}$$

For the present plane problem, the deformation gradient takes on the form

$$F_{ij} = \begin{pmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{pmatrix}. \tag{8}$$

The non-zero components of the deformation gradient are

$$\begin{aligned} F_{11} &= R^{\alpha-1}(\alpha f_1\cos\Theta - f_1'\sin\Theta) = R^{\alpha-1}h_{11}(\Theta), \\ F_{12} &= R^{\alpha-1}(\alpha f_1\sin\Theta + f_1'\cos\Theta) = R^{\alpha-1}h_{12}(\Theta), \\ F_{21} &= R^{\beta-1}(\beta f_2\cos\Theta - f_2'\sin\Theta) = R^{\beta-1}h_{21}(\Theta), \\ F_{22} &= R^{\beta-1}(\beta f_2\sin\Theta + f_2'\cos\Theta) = R^{\beta-1}h_{22}(\Theta), \\ F_{33} &= F_{33}(Z) \leqslant 1, \end{aligned} \tag{9}$$