ARTICLE IN PRESS

International Journal of Solids and Structures xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Shakedown reduced kinematic formulation, separated collapse modes, and numerical implementation

T.D. Tran [a], C.V. Le [b], D.C. Pham [c,*], H. Nguyen-Xuan [d,e]

[a] Faculty of Construction & Electricity, Ho Chi Minh City Open University, 97 Vo Van Tan Street, Ho Chi Minh City, Viet Nam
[b] Department of Civil Engineering, International University – VNU HCMC, Viet Nam
[c] Institute of Mechanics – VAST, 264 Doi Can, Hanoi, Viet Nam
[d] Duy Tan University, Da Nang, Viet Nam
[e] Department of Computational Engineering, Vietnamese-German University, Binh Duong New City, Viet Nam

A R T I C L E   I N F O

Article history:
Received 16 December 2013
Received in revised form 14 April 2014
Available online xxxx

Keywords:
Plastic collapse
Shakedown
Elastic–plastic material
Cyclic loading
Reduced kinematic formulation
Incremental plasticity
Alternating plasticity
Second-order cone programming

A B S T R A C T

Our shakedown reduced kinematic formulation is developed to solve some typical plane stress problems, using finite element method. Whenever the comparisons are available, our results agree with the available ones in the literature. The advantage of our approach is its simplicity, computational effectiveness, and the separation of collapse modes for possible different treatments. Second-order cone programming developed for kinematic plastic limit analysis is effectively implemented to study the incremental plasticity collapse mode. The approach is ready to be used to solve general shakedown problems, including those for elastic–plastic kinematic hardening materials and under dynamic loading.

1. Introduction

Shakedown analysis determines the load limits on the loading cycles, under which a structure should be safe (i.e. it would behave elastically after some possible initial limited plastic dissipation), while over the limits the structure would collapse incrementally or fail by alternating plasticity (Melan, 1938; Koiter, 1963; Gokhfeld, 1966; Debordes, 1977; König, 1987; Bree, 1989; Pham and Stumpf, 1994; Pham, 2000b, 2003b, 2008, 2013). Melan's static and Koiter's kinematic shakedown theorems are generalizations of the respective plastic limit ones (Gvozdev, 1960; Drucker et al., 1951; Hill, 1951), from the instantaneous collapse considerations to those over loading processes. Original shakedown theorems are restricted to quasi-static loading processes, however latter have been extended for the larger class of dynamic problems (Gavarini, 1969; Ho, 1972; Corradi and Maier, 1974; Ceradini, 1980; Pham, 1992, 2000a, 2003a, 2010; Corigliano et al., 1995). Preliminarily restricted to elastic-perfectly plastic bodies, the shakedown theorems have been developed for more general kinematic hardening materials (Maier, 1972; Ponter, 1975; König, 1987; Weichert and Gross-Weege, 1988; Polizzotto et al., 1991; Stein and Huang, 1994; Fuschi, 1999; Pham and Weichert, 2001; Weichert and Maier, 2002; Bousshine et al., 2003; Pham, 2007, 2008, 2013; Simon, 2013; . . .).

Implementing the shakedown theorems in applications, one faces the difficulty of solving nonlinear optimization problems for structures with complex geometries and under complicated loading programs (Belytschko, 1972; Corradi and Zavelani, 1974; Zouain et al., 2002; Magoariec et al., 2004; Vu et al., 2004a; Liu et al., 2005; Garcea et al., 2005; Chen et al., 2008; Tran et al., 2010). Various iterative optimization algorithms have been developed to provide solution of such the non-linear programming (Zouain et al., 2002; Vu et al., 2004b; Garcea et al., 2005; Li and Yu, 2006). However, these methods can tackle problems with a moderate number of variables, and hence it is still desirable to develop an alternative solution procedure that can solve large-scale shakedown analysis problems in engineering practices. Shakedown analysis is a generation of limit analysis, and hence optimization algorithms initially developed for the latter can usually be extended to former in a relatively straightforward manner. In the context of limit analysis, a primal–dual interior-point method proposed in Andersen et al. (2001, 2003)) has been proved to be one of the most robust and efficient algorithms in treating

* Corresponding author. Tel.: +84 4 37 628 125; fax: +84 437622039.
E-mail address: pdchinh@imech.ac.vn (D.C. Pham).

http://dx.doi.org/10.1016/j.ijsolstr.2014.04.016
0020-7683/© 2014 Elsevier Ltd. All rights reserved.

such the large-scale non-linear optimization problem (Ciria et al., 2008; Munoz et al., 2009; Le et al., 2009, 2010, 2012). The algorithm has been extended to both static and kinematic shakedown analysis problems (Le et al., 2010; Bisbos et al., 2005; Makrodimopoulos, 2006; Weichert and Simon, 2012).

From the original Koiter's shakedown kinematic theorem, reduced shakedown kinematic formulations have been deduced with separated collapse modes and applied to various simple typical structures (of plastic limit analysis), yielding certain interesting semi-analytical results (Pham, 1992, 2000a,b, 2003a,b, 2008, 2010, 2013; Pham and Stumpf, 1994;...). The purpose of this research is to develop this approach with numerical finite element implementation for more complex engineering structures. The primal–dual interior-point algorithm will be developed with the reduced shakedown kinematic formulation, making the use of the optimization method in direct manner. It is worth noting that with the use of the reduced kinematic formulation the number of kinematic variables in the resulting optimization is kept to a minimum. This is because there is no need to approximate displacement fields at every vertex of a polyhedral load domain as required in the original kinematic formulation. Furthermore, the reduced kinematic formulation is able to produce independently the incremental and alternative modes, leading to possible different treatments of the modes in analysis of structures.

The layout of the paper is as follows. The next section recalls kinematic formulations including both unified and reduced kinematic shakedown ones. In Section 3, the kinematic formulation is discretized using finite element method, and an optimization strategy based on second-order cone programming is described. Numerical examples are provided in Section 4 to illustrate the performance of the proposed solution strategy, and non-shakedown modes for various loading domains are also shown. The conclusion completes the paper.

2. Shakedown kinematic formulations

Let $\boldsymbol{\sigma}^e(\mathbf{x},t)$ denote the fictitious elastic stress response of the body V to external agencies over a period of time ($\mathbf{x}\in V,\ t\in[0,T]$) under the assumption of perfectly elastic behavior, called a loading process (history). The actions of all kinds of external agencies upon V can be expressed explicitly through $\boldsymbol{\sigma}^e$. At every point $\mathbf{x}\in V$, the elastic stress response $\boldsymbol{\sigma}^e(\mathbf{x},t)$ is confined to a bounded time-independent domain with prescribed limits in the stress space, called a local loading domain $\mathcal{L}_x$. As a field over V, $\boldsymbol{\sigma}^e(\mathbf{x},t)$ belongs to the time-independent global loading domain $\mathcal{L}$:

$$\mathcal{L}=\{\boldsymbol{\sigma}^e|\boldsymbol{\sigma}^e(\mathbf{x},t)\in\mathcal{L}_x,\quad \mathbf{x}\in V,\quad t\in[0,T]\}. \tag{1}$$

In the spirit of classical shakedown analysis, the bounded loading domain $\mathcal{L}$, instead of a particular loading history $\boldsymbol{\sigma}^e(\mathbf{x},t)$, is given a priori. Shakedown of a body in $\mathcal{L}$ means it shakes down for all possible loading histories $\boldsymbol{\sigma}^e(\mathbf{x},t)\in\mathcal{L}$.

$\mathcal{A}$ signifies the set of compatible-end-cycle (deviatoric) plastic strain rate fields $\mathbf{e}^p$ over time cycles $0\leqslant t\leqslant T$:

$$\mathcal{A}=\left\{\mathbf{e}^p|\boldsymbol{\varepsilon}^p=\int_0^T\mathbf{e}^p dt\in\mathcal{C}\right\}, \tag{2}$$

where $\mathcal{C}$ is the set of strain fields that are both deviatoric and compatible on V. Let k_s be the shakedown safety factor: at $k_s>1$ the structure will shake down, while it will not at $k_s<1$, and $k_s=1$ defines the boundary of the shakedown domain. Koiter's shakedown kinematic theorem can be stated as (Koiter, 1963; Pham, 2003a):

$$k_s^{-1}=\sup_{\mathbf{e}^p\in\mathcal{A};\boldsymbol{\sigma}^e\in\mathcal{L}}\frac{\int_0^T dt\int_V\boldsymbol{\sigma}^e:\mathbf{e}^p dV}{\int_0^T dt\int_V D(\mathbf{e}^p)dV}, \tag{3}$$

where $D(\mathbf{e}^p)=\boldsymbol{\sigma}:\mathbf{e}^p$ is the dissipation function determined by the yield stress σ_Y and the respective yield criterion; e.g. for a Mises material we have

$$D(\mathbf{e}^p)=\sqrt{2/3}\,\sigma_Y(\mathbf{e}^p:\mathbf{e}^p)^{1/2}. \tag{4}$$

Alternatively, (3) can also be presented as

$$k_s=\inf_{\mathbf{e}^p\in\mathcal{A};\boldsymbol{\sigma}^e\in\mathcal{L}}\frac{\int_0^T dt\int_V D(\mathbf{e}^p)dV}{\int_0^T dt\int_V\boldsymbol{\sigma}^e:\mathbf{e}^p dV}, \tag{5}$$

but with the implicit condition that $\int_0^T dt\int_V\boldsymbol{\sigma}^e:\mathbf{e}^p dV>0$ (otherwise the expression $\inf(\cdot)$ should be trivial $-\infty$, which is physically meaningless).

With the use of mathematical programming theory, the upper bound shakedown theorem can be re-expressed in the form of an optimization problem as follows

$$k_s^+=\min\int_0^T dt\int_V D(e_{ij}^p)dV$$
$$\text{s.t}\begin{cases}\int_0^T dt\int_V\sigma_{ij}^e e_{ij}^p dV=1\\ \Delta\varepsilon_{ij}^p=\int_0^T e_{ij}^p dt=\frac{1}{2}(\Delta u_{i,j}+\Delta u_{j,i}) & \text{on } V,\\ \Delta u_i=\int_0^T\dot{u}_i dt & \text{on } V,\\ \Delta u_i=0 & \text{on } \partial V_u,\end{cases} \tag{6}$$

where k_s^+ is the upper bound on the actual shakedown load multiplier, $\Delta\varepsilon_{ij}^p$ is the admissible cycle of plastic strain fields corresponding to a cycle of displacement fields Δu_i, and the constrained boundary ∂V_u is fixed. Note that at each instant during the time cycle t, the plastic strain rates e_{ij}^p may be not compatible, but the plastic strain accumulated over the cycle $\Delta\varepsilon_{ij}$ must be compatible.

In order to perform numerical shakedown analysis of structures, the time integration in problem (6) must be removed because the evaluation of plastic strains over a loading cycle would be difficult. Based on the two convex-cycle theorems presented in König (1987), the problem (6) can be expressed as

$$k_s^+=\min\sum_{k=1}^M\int_V D(\mathbf{e}_k^p)dV$$
$$\text{s.t}\begin{cases}\sum_{k=1}^M\int_V\boldsymbol{\sigma}_k^e:\mathbf{e}_k^p dV=1\\ \Delta\boldsymbol{\varepsilon}^p=\sum_{k=1}^M\mathbf{e}_k^p & \text{on } V,\\ \Delta\mathbf{u}=\mathbf{0} & \text{on } \partial V_u,\end{cases} \tag{7}$$

where $M=2^N$ is the number of vertices of the convex polyhedral load domain $\mathcal{L}$; N is the number of variable loads.

The so-called unified optimization problem (7) can provide a shakedown load multiplier that is the smaller one of incremental plasticity limit (ratchetting or progressive deformation limit) and alternating plasticity limit (low-cycle fatigue or plastic shakedown limit), and it has been solved numerically using various discretization method and optimization algorithms.

From Koiter's shakedown kinematic theorem (3), Pham (1992) and Pham and Stumpf (1994) have deduced the much simpler reduced shakedown kinematic formulation

$$k_s^{-1}\geqslant k_{sr}^{-1}=\max\{I,A\}, \tag{8}$$

where

$$I=\sup_{\boldsymbol{\sigma}^e\in\mathcal{L};\boldsymbol{\varepsilon}^p\in\mathcal{C}}\frac{\int_V\max_{t_x}[\boldsymbol{\sigma}^e(\mathbf{x},t_x):\boldsymbol{\varepsilon}^p(\mathbf{x})]dV}{\int_V D(\boldsymbol{\varepsilon}^p)dV}, \tag{9}$$

$$A=\sup_{\mathbf{x}\in V;\boldsymbol{\sigma}^e\in\mathcal{L};\hat{\boldsymbol{\varepsilon}}^p;t_1,t_2}\frac{[\boldsymbol{\sigma}^e(\mathbf{x},t_1)-\boldsymbol{\sigma}^e(\mathbf{x},t_2)]:\hat{\boldsymbol{\varepsilon}}^p(\mathbf{x})}{2D(\hat{\boldsymbol{\varepsilon}}^p)} \tag{10}$$