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Lattice with long-range interaction of power-law type for fractional non-local elasticity

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ABSTRACT

Lattice models with long-range interactions of power-law type are suggested as a new type of microscopic model for fractional non-local elasticity. Using the transform operation, we map the lattice equations into continuum equation with Riesz derivatives of non-integer orders. The continuum equations that are obtained from the lattice model describe fractional generalization of non-local elasticity models. Particular solutions and correspondent asymptotic of the fractional differential equations for displacement fields are suggested for the static case.

1. Introduction

Lattice with long-range interaction is a subject of investigations in different areas of mechanics and physics (see for example Kröner, 1967; Eringen and Kim, 1977; Ostoja-Starzewski, 2002; Luo and Afraimovich, 2010; Tarasov, 2011; Dyson, 1971; Frohlich et al., 1978; Nakano and Takahashi, 1995; Campa et al., 2009). The long-range interactions have been studied in discrete systems as well as in their continuous analogs. As it was shown in Tarasov (2006b,a) (see also Tarasov and Zaslavsky, 2006; Tarasov, 2011), the continuum equations with derivatives of non-integer orders can be directly connected to lattice models with long-range interactions of power law type.

The theory of derivatives and integrals of non-integer orders (Samko et al., 1993; Kilbas et al., 2006) allow us to investigate the behavior of materials and media that are characterized by nonlocality of power-law type. Fractional calculus has a wide application in mechanics and physics (for example see Carpinteri and Mainardi, 1997; Hilfer, 2000; Sabatier et al., 2007; Mainardi, 2010; Luo and Afraimovich, 2010; Tarasov, 2011, 2013a; Klafter et al., 2011). The fractional calculus allows us to formulate a fractional generalization of non-local elasticity models in two forms: the fractional gradient elasticity models (weak power-law nonlocality) and the fractional integral non-local models (strong power-law non-locality). Fractional models of non-local elasticity and some microscopic models are considered in different articles (see for example Lazopoulos, 2006; Cottone et al., 2009; Carpinteri et al., 2009a,b, 2011; Di Paola and Zingales, 2008, 2009, 2011; Di Paola et al., 2010, 2014; Tarasov, 2014, 2013, 2014a). Elastic waves in nonlocal continua modeled by a fractional calculus approach are considered in Cottone et al. (2009), Atanackovic and Stankovic (2009), Zingales (2011), Sapora et al. (2013) and Challamel et al. (2013). In Tarasov (2014) and Tarasov (2013) a general approach to describe lattice model with power-law spatial dispersion for fractional elasticity has been proposed. This approach can be used for different type of interaction of lattice particles. Therefore explicit forms of the long-range interactions are not considered in Tarasov (2014, 2013). In Tarasov (2014a) a model of lattice with long-range interaction of Grünwald–Letnikov–Riesz type has been suggested to describe fractional gradient and integral elasticity of continuum. In this paper we focus on the lattice models with long-range interaction of power-law type as new type of microscopic models for fractional generalization of elasticity theory. We suggest lattice models with power-law long-range interaction as microscopic model of fractional non-local continuum. The equations for displacement field are directly derived from the suggested lattice models by the methods of Tarasov (2006b,a). The suggested generalization of the elasticity equations contains the fractional Laplacian in the Riesz's form (Kilbas et al., 2006). We demonstrate a connection between the dynamics of lattice system of particles with long-range interactions and the fractional continuum equations by using the transform operation suggested in Tarasov (2006b,a). We show how the continuous limit for the lattice with long-range interactions of power-law type gives the continuum equation of the fractional elasticity. We get particular

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solutions of the fractional differential elasticity equations for some special cases.

2. Equations of lattice model

As a microscopic model, we use unbounded homogeneous lattices, such that all particles are displaced from its equilibrium positions in one direction, and the displacement of particle is described by a scalar field. We consider one-dimensional lattice system of interacting particles. The equations of motion for particles are

$$M\frac{d^2u_n(t)}{dt^2} = g_2 \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_2(n,m)\,(u_n - u_m) + g_\alpha \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_\alpha(n,m)\,(u_n - u_m) + F(n), \tag{1}$$

where $u_n(t)$ are displacements from the equilibrium, g_2 and g_α are the coupling constants of particle interactions, and the terms $F(n)$ characterize an interaction of the particles with the external on-site force. For simplicity, we assume that all particles have the same mass M. The function $K_2(n,m)$ describes the nearest-neighbor interaction with coupling constant $g_2 = K$, which is the spring stiffness. The function $K_\alpha(n,m)$ describes the long-range interaction with a coupling constant g_α. For a simple case each particle can be considered an inversion center and

$$K_\alpha(n,m) = K_\alpha(|n-m|).$$

Equations of motion (1) have the invariance with respect to its displacement of lattice as a whole in case of absence of external forces. It should be noted that the non-invariant terms lead to the divergences in the continuous limit (Tarasov, 2011).

Using the approach suggested in Tarasov (2006a,b, 2011), we can consider a set operations that transforms the lattice equations for $u_n(t)$ into continuum equation for displacement field $u(x,t)$. We assume that $u_n(t)$ are Fourier coefficients of the field $\hat{u}(k,t)$ on $[-k_0/2, k_0/2]$ that is described by the equations

$$u_n(t) = \frac{1}{k_0}\int_{-k_0/2}^{+k_0/2} dk\ \hat{u}(k,t)\, e^{ikx_n} = \mathcal{F}_\Delta^{-1}\{\hat{u}(k,t)\}, \tag{2}$$

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} u_n(t)\, e^{-ikx_n} = \mathcal{F}_\Delta\{u_n(t)\}, \tag{3}$$

where $x_n = nd$ and $d = 2\pi/k_0$ is distance between equilibrium positions of the lattice particles. Eqs. (3) and (2) are the basis for the Fourier series transform $\mathcal{F}_\Delta$ and the inverse Fourier series transform $\mathcal{F}_\Delta^{-1}$.

The Fourier transform can be derived from (3) and (2) in the limit as $d \to 0$ ($k_0 \to \infty$). In this limit the sum is transformed into an integral, and Eqs. (2) and (3) become

$$\tilde{u}(k,t) = \int_{-\infty}^{+\infty} dx\ e^{-ikx} u(x,t) = \mathcal{F}\{u(x,t)\}, \tag{4}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ e^{ikx}\tilde{u}(k,t) = \mathcal{F}^{-1}\{\tilde{u}(k,t)\}. \tag{5}$$

Here we use the lattice function

$$u_n(t) = \frac{2\pi}{k_0} u(x_n, t)$$

with continuous function $u(x,t)$, where $x_n = nd = (2\pi n)/k_0 \to x$. We assume that $\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t)$, where $\mathcal{L}$ denotes the passage to the limit $d \to 0$ ($k_0 \to \infty$), i.e. the function $\tilde{u}(k,t)$ can be derived from $\hat{u}(k,t)$ in the limit $d \to 0$. Note that $\tilde{u}(k,t)$ is a Fourier transform of the field $u(x,t)$. The function $\hat{u}(k,t)$ is a Fourier series transform of $u_n(t)$, where we can use $u_n(t) = (2\pi/k_0)u(nd,t)$.

We can state that a lattice model transforms into continuum model by the combination $\mathcal{F}^{-1}\mathcal{L}\,\mathcal{F}_\Delta$ of the following operation (Tarasov, 2006a,b):

The Fourier series transform:

$$\mathcal{F}_\Delta: \quad u_n(t) \to \mathcal{F}_\Delta\{u_n(t)\} = \hat{u}(k,t). \tag{6}$$

The passage to the limit $d \to 0$:

$$\mathcal{L} = \lim_{d\to 0}: \quad \hat{u}(k,t) \to \mathcal{L}\{\hat{u}(k,t)\} = \tilde{u}(k,t). \tag{7}$$

The inverse Fourier transform:

$$\mathcal{F}^{-1}: \quad \tilde{u}(k,t) \to \mathcal{F}^{-1}\{\tilde{u}(k,t)\} = u(x,t). \tag{8}$$

These operations allow us to get continuum equations from the lattice equations (Tarasov, 2006a,b, 2011).

3. Lattice with nearest-neighbor interaction

Let us consider the lattice with nearest-neighbor interaction that is described by (1), where $K_\alpha(n-m) = 0$, and

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_2(n,m)\, u_m(t) = u_{n+1}(t) - 2u_n(t) + u_{n-1}(t), \tag{9}$$

where the term $K_2(n,m)$ describes the nearest-neighbor interaction. Let us derive the usual elastic equation from the lattice model with the nearest-neighbor interaction with coupling constant $g_2 = K$, which is the spring stiffness. The following statement (Tarasov, 2006a,b, 2011) gives for this lattice model with the nearest-neighbor interaction the corresponding continuum equation in the limit $d \to 0$.

Proposition 1. *In the continuous limit ($d \to 0$) the lattice equations of motion*

$$M\frac{d^2u_n(t)}{dt^2} = K \cdot (u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + F(n) \tag{10}$$

are transformed by the combination $\mathcal{F}^{-1}\mathcal{L}\,\mathcal{F}_\Delta$ of the operations (6)–(8) into the continuum equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = C_e^2\,\Delta u(x,t) + \frac{1}{\rho} f(x), \tag{11}$$

where

$$\rho = \frac{M}{Ad}, \quad C_e^2 = \frac{E}{\rho} = \frac{K d^2}{M}, \quad E = \frac{K d}{A} \tag{12}$$

and C_e^2 is a finite parameter, A is the cross-section area of the medium, E is the Youngs modulus, and $f(x) = F(x)/(Ad)$ is the force density.

A detailed proof of Proposition 1 is given in Appendix A.

As a result, we prove that lattice Eq. (10) in the limit $d \to 0$ give the continuum equation with the Laplacian (see also Tarasov, 2014b). Note that this result can be derived by methods described in Section 8 of Maslov (1976), where the relation

$$\exp i\left(-i d\frac{\partial}{\partial x}\right) u(x,t) = u(x+d,t)$$

and the representation of (10) by pseudo-differential equation are used.