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On an inverse problem for inhomogeneous thermoelastic rod

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ABSTRACT

In recent years, different fields of engineering have been increasingly incorporating functionally graded materials with variable physical properties that significantly improve a quality of elements of designs. The efficiency of practical application of thermoelastic inhomogeneous materials depends on knowledge of exact laws of heterogeneity, and to define them it is necessary to solve coefficient inverse problems of thermoelasticity.

In the present research a scheme of solving the inverse problem for an inhomogeneous thermoelastic rod is presented. Two statements of the inverse problem are considered: in the Laplace transform space and in the actual space. The direct problem solving is reduced to a system of the Fredholm integral equations of the 2nd kind in the Laplace transform space and an inversion of the solutions obtained on the basis of the theory of residues. The inverse problem solving is reduced to an iterative procedure, at its each step it is necessary to solve the Fredholm integral equation of the 1st kind; to solve it the Tikhonov method is used. Specific examples of a reconstruction of variable characteristics required are given.

1. Introduction

For many years, layered composites have been widely used as a coating for elements of structures operating in high temperature environment (such as airplanes and spaceships, gas turbine blades, cutting tools, implants in biomedical industry, etc.) as they have been providing the structure's mechanical and thermal properties required. However, the jumps of material properties through the interface between discrete materials may cause large stress concentration and formation of plastic deformation or cracking.

In recent years, functionally graded materials have been used as an alternative to layered composites (Aboudi et al., 1995; Lee et al., 1996; Suresh and Mortensen, 1998; Wetherhold et al., 1996) to avoid material properties jumps through the interface due to their continual change. In this case, the thermomechanical characteristics are not constants but some functions of spatial coordinates, i.e. the material acquires the spatial heterogeneity. Such inhomogeneous structure may be obtained not only within the manufacturing process, but also during an operation under radiation, strong magnetic fields, and heavy temperature drops.

It is almost impossible to predict changes in the structure of materials caused by external actions. The efficiency of practical application of thermoelastic inhomogeneous materials depends on knowing exact laws of heterogeneity. The problem of finding the thermomechanical characteristics of inhomogeneous bodies is the coefficient inverse problem of thermoelasticity.

To date, there is already a considerable experience of investigation inverse problems. The general methods of solving inverse problems are presented in the monographs and papers (Alifanov et al., 1988; Denisov, 1994; Isakov, 2005; Kabanikhin, 2009; Vatulyan, 2007; Gockenbach et al., 2008; Jadamba et al., 2011), etc. But still there is a lack of researches of the coefficient inverse problems of thermoelasticity (Apbasov and Yahno, 1986; Lomazov, 2002; Lukasievicz et al., 1996). However, some specific problems concerned with finding variable coefficients of operators of thermal conductivity (Alifanov et al., 1988; Dimitriau, 2001; Hao, 1998; Isakov and Bindermann, 2000; Pobedrya et al., 2008; Xu et al., 2002) and of the elasticity theory (Alekseev, 1967; Belishev and Blagovecshenskey, 1999; Chen and Gockenbach, 2007; Jadamba et al., 2008; Kabanikhin, 1988; Rakesh, 1993; Vatulyan, 2010; Yakhno, 1990) are separately studied good enough. One of the basic approaches to a solving of inverse problems of heat conduction is its reduction to minimize non-quadratic residual functional in a finite dimensional subspace (Alifanov et al., 1988; Kabanikhin et al., 2008; Pobedrya et al., 2008). It is necessary to use iterative processes requiring the calculation of the functional gradient at each step. There is an extensive theoretical foundation for gradient minimization techniques (Alifanov et al., 1988; Hao, 1998). However, the shortcomings of such techniques are a strong influence of a choice of initial approximation on a convergence of the iteration process, and requirements to the objective function. In addition, with increasing number of unknowns delivering the minimum of

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the objective function, the amount of calculations significantly increases.

When solving inverse problems of the theory of elasticity, the most commonly used method are the method of Volterra operators (Yakhno, 1990), the method of inversion of the difference scheme (Kabanikhin, 1988), the linearization method (Alekseev, 1967), and the boundary control method (Belishev and Blagovecshenskey, 1999). At the same time for a number of modern materials, when solving the inverse problems, it is necessary to take into account the coupling of thermal and mechanical fields. The inverse problems of thermoelasticity for inhomogeneous bodies are scarcely investigated and mainly limited by weakly inhomogeneous materials (Lomazov, 2002) due to the difficulties in the construction of nonlinear operator relations that bind the desired and the measured (during an experiment) functions. However, in some papers (Vatulyan, 2007; Dudarev and Vatulyan, 2011; Nedin and Vatulyan, 2011; Nedin and Vatulyan, 2013a; Nedin and Vatulyan, 2013b) devoted to the inverse problems of the mechanics of related fields, this difficulty was overcome with the help of generalized reciprocal relations. At that the linearized Fredholm integral equations of the 1st kind were obtained to find the corrections of the coefficients recovered.

In the present paper the formulations and solutions of the coefficient inverse problem for an inhomogeneous rod are described in case of an arbitrary coupling parameter. After applying the Laplace transformation, the direct problem was reduced to solving a system of the Fredholm integral equations of the 2nd kind with respect to the transforms of temperature and pressure, and finding the actual space on the basis of the theory of residues. On the basis of the generalized reciprocity relation and the linearization method, the inverse problem was reduced to stepwise solving of the Fredholm integral equation of the 1st kind. A series of computational experiments was conducted for exact and noisy input data. The recommendations for a practical employment of the approach proposed are given.

2. Statement of the inverse problem

Let us consider a problem of longitudinal oscillations of the inhomogeneous thermoelastic rod of length l rigidly fixed at the end $x = 0$, and distinguish two ways of oscillations excitation: the thermal way and the mechanical one.

In case of the excitation of oscillations under the action of the heat flow $Q = q_0\varphi(t)$ applied to the end $x = l$ the initial-boundary value problem takes the following form (Nowacki, 1970; Vatulyan and Nesterov, 2012):

$$\frac{\partial \sigma_x}{\partial x} = \rho(x)\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\sigma_x = E(x)\frac{\partial u}{\partial x} - \gamma(x)\theta, \tag{2}$$

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial \theta}{\partial x}\right) = c_\varepsilon(x)\frac{\partial \theta}{\partial t} + T_0\gamma(x)\frac{\partial^2 u}{\partial x \partial t}, \tag{3}$$

$$\theta(0,t) = u(0,t) = 0, \quad -k(l)\frac{\partial \theta}{\partial x}(l,t) = q_0\varphi(t), \quad \sigma_x(l,t) = 0, \tag{4}$$

$$\theta(x,0) = u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0, \tag{5}$$

If the rod is oscillated by the force $F = p_0\lambda(t)$ applied to the end $x = l$ then the boundary conditions (4) in the problem (1)–(5) will take another form:

$$\frac{\partial \theta}{\partial x}(l,t) = 0, \quad \sigma_x(l,t) = p_0\lambda(t). \tag{6}$$

The inverse problem is to determine one of the thermomechanical characteristics of the rod (the specific volumetric heat capacity $c_\varepsilon(x)$, the thermal conductivity $k(x)$, the rod's density $\rho(x)$, the Young modulus $E(x)$, the coefficient of thermal stress $\gamma(x)$) when knowing the rest characteristics of (1)–(5) on the basis of some additional data at the boundary.

In case of the thermal loading of the rod the temperature increment at its end is used as the additional data:

$$\theta(l,t) = f(t), \quad t \in [T_1, T_2] \tag{7}$$

and in case of the mechanical loading the displacement at the rod's end is used:

$$u(l,t) = g(t), \quad t \in [T_3, T_4]. \tag{8}$$

If the temperature and the displacement are known at any given time then the inverse problem may be formulated in the Laplace transform space.

In this case the additional information is

$$\theta(l,p) = \tilde{f}(p), \quad p \in [0,\infty), \tag{9}$$

$$u(l,p) = \tilde{g}(p), \quad p \in [0,\infty). \tag{10}$$

3. Solving the direct problem for an inhomogeneous thermoelastic rod

The direct problem on the vibration of a thermoelastic rod with arbitrary laws of variation of coefficients of differential operators can be solved only numerically.

Let us rewrite (1)–(5) in a dimensionless form. To do this we introduce the following parameters and variables: $z = \frac{x}{l}$, $z \in [0,1]$, $\bar{k}(z) = \frac{k(zl)}{k_0}$, $\bar{c}(z) = \frac{c(zl)}{c_0}$, $\bar{\rho}(z) = \frac{\rho(zl)}{\rho_0}$, $\bar{E}(z) = \frac{E(zl)}{E_0}$, $\bar{\gamma}(z) = \frac{\gamma(zl)}{\gamma_0}$, $t_1 = \frac{l^2 c_0}{k_0}$, $t_2 = l\sqrt{\frac{\rho_0}{E_0}}$, $\tau_1 = \frac{t}{t_1}$, $W_1 = \frac{\gamma_0\theta}{E_0}$, $U_1 = \frac{u}{l}$, $\Omega_1 = \frac{\sigma_x}{E_0}$, $\delta = \frac{\gamma_0^2 T_0}{c_0 E_0}$, $\omega = \frac{q_0\gamma_0 l}{k_0 E_0}$, $\varepsilon = \frac{t_2}{t_1} = \frac{k_0}{c_0 l}\sqrt{\frac{\rho_0}{E_0}}$, $k_0 = \max_{x\in[0,l]} k(x)$, $c_0 = \max_{x\in[0,l]} c(x)$, $E_0 = \max_{x\in[0,l]} E(x)$, $\rho_0 = \max_{x\in[0,l]}\rho(x)$, $\gamma_0 = \max_{x\in[0,l]}\gamma(x)$.

Here δ is the dimensionless coupling parameter, ε is the ratio of the characteristic time of the sound vibration t_2 to the time of the thermal vibration t_1.

Hence, the boundary-value problem (1)–(5) takes the form:

$$\frac{\partial \Omega_1}{\partial z} = \varepsilon^2\bar{\rho}(z)\frac{\partial^2 U_1}{\partial \tau_1^2}, \tag{11}$$

$$\Omega_1 = \bar{E}(z)\frac{\partial U_1}{\partial z} - \bar{\gamma}(z)W_1, \tag{12}$$

$$\frac{\partial}{\partial z}\left(\bar{k}(z)\frac{\partial W_1}{\partial z}\right) = \bar{c}_\varepsilon(z)\frac{\partial W_1}{\partial \tau_1} + \delta\bar{\gamma}(z)\frac{\partial^2 U_1}{\partial z \partial \tau_1}, \tag{13}$$

$$U_1(0,\tau_1) = W_1(0,\tau_1) = 0, \quad -\bar{k}(1)\frac{\partial W_1}{\partial z}(1,\tau_1) = \omega\varphi(\tau_1), \quad \Omega_1(1,\tau_1) = 0, \tag{14}$$

$$W_1(z,0) = U_1(z,0) = \frac{\partial U_1}{\partial \tau_1}(z,0) = 0. \tag{15}$$

In case of the excitation of longitudinal oscillations under the action of mechanical load $p_0\lambda(t)$ we may formulate the dimensionless problem in a same way as previously except the following difference: $\mu = \frac{p_0}{E_0}$, $\tau_2 = \frac{t}{t_2}$, $W_2 = \frac{\gamma_0\theta}{E_0}$, $U_2 = \frac{u}{l}$, $\Omega_2 = \frac{\sigma_x}{E_0}$.

In this case the dimensionless boundary conditions take the form:

$$W_2(0,\tau_2) = U_2(0,\tau_2) = 0, \quad \frac{\partial W_2}{\partial z}(1,\tau_2) = 0, \quad \Omega_2(1,\tau_2) = \mu\lambda(\tau_2). \tag{16}$$