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A numerical study of elastic bodies that are described by constitutive equations that exhibit limited strains

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ABSTRACT

Recently, a very general and novel class of implicit bodies has been developed to describe the elastic response of solids. It contains as a special subclass the classical Cauchy and Green elastic bodies. Within the class of such bodies, one can obtain through a rigorous approximation, constitutive relations for the linearized strain as a nonlinear function of the stress. Such an approximation is not possible within classical theories of Cauchy and Green elasticity, where the process of linearization will only lead to the classical linearized elastic body.

In this paper, we study numerically the states of stress and strain in a finite rectangular plate with an elliptic hole and a stepped flat tension bar with shoulder fillets, within the context of the new class of models for elastic bodies that guarantees that the linearized strain would stay bounded and limited below a value that can be fixed a priori, thereby guaranteeing the validity of the use of the model. This is in contrast to the classical linearized elastic model, wherein the strains can become large enough in the body leading to an obvious inconsistency.

1. Introduction

Recently, a new class of implicit constitutive relations was introduced to describe the response of elastic bodies (see Rajagopal, 2003, 2007, 2011b, Rajagopal and Srinivasa, 2007, 2009 and Bustamante, 2009). This new class includes the explicit theories of Cauchy elasticity and Green elasticity as special subclasses. The advantages that such models provide over the classical models are detailed in several papers (Rajagopal, 2011b,a; Bustamante and Rajagopal, 2010) and hence we shall not repeat them here. Suffice it is to say that very important problems such as the problem of fracture, which has defied a proper consistent explanation without resorting to ad hoc procedures (see Rajagopal and Walton, 2011, Kulvait et al., 2013, Ortiz et al., 2012, and Bulíček et al., 2013 with regard to how the problem is dealt within the context of the new class of models) and the modeling of certain phenomena exhibited by soft material that has hitherto defied explanation within the context of classical models (see the discussion in Freed and Einstein (2013a,b); Freed et al., 2013) are some examples of the potential of the new class of implicit constitutive relations. The class of implicit models has also been extended to develop models to describe the electroelastic response of bodies and it has been able to describe phenomena that have thus far been impossible to explain within the context of classical electroelastic theories (see Bustamante and Rajagopal, 2013b,a).

Another special subclass of the implicit models for elasticity introduced by Rajagopal (2003) is explicit models for the stretch in terms of the stress (see Rajagopal, 2007, 2011b) and its linearization that leads to an explicit nonlinear expression for the linearized strain in terms of the stress. The latter class of models is impossible within the context of classical theories of elasticity and this paper is concerned with a study of such models (see the model defined through (15)). When one is concerned with constitutive relations for the Cauchy–Green stretch or the linearized strain in terms of the stress, one does not have the luxury of substituting the expression for the stress in terms of the displacement gradient into the balance of linear momentum and obtaining a partial differential equation for the displacement field. Instead, the constitutive relation, the balance of linear momentum, and whatever other balance laws are relevant, need to be solved simultaneously; hence the stress and displacement fields are both unknowns that need to be solved for. This system of coupled nonlinear partial differential equations is far more daunting than the much simplified system that is obtained when an explicit expression for the stress in terms of the displacement gradient can be substituted into the balance of linear momentum. In this study, we are concerned with an explicit expression for the linearized strain in terms of the stress and hence concerned with the more complex system of coupled partial differential equations.

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Though the system is complicated in that the number of equations to be solved is larger, the order of the equations within the purview of the new framework is a system of lower order equations and thus from the point of view of numerical analysis provides some advantages.

Most of the studies until recently have been concerned with special boundary value problems, in infinite domains, wherein semi-inverse assumptions are made to reduce the problem to the study of much simplified governing equations, which in most instances is a system of ordinary differential equations. In this paper, we shall study problems in finite domains and as it is unlikely that we can simplify the problem to obtain simpler ordinary differential equations, we shall have to study the problem numerically. We will study two problems, that of the stress concentration due to the presence of an elliptic hole and that of the stress concentration in a stepped flat tension bar with shoulder fillets, within the context of the new class of models. The first of the two problems has relevance to the problem of stress concentration due to a crack as such a situation can be achieved by taking the limit of the ratio of the minor axis to the major axis of the elliptic hole to tend to zero.

The organization of the paper is as follows. In Section 2, we introduce the basic kinematics, document the general implicit constitutive relation between the stress and the stretch for isotropic bodies, and derive a special constitutive relation for the linearized strain in terms of the stress under the assumption that the displacement gradient is small. We then record some special constitutive expressions for the linearized strain in terms of the stress and develop the system of governing equations that need to be solved. In Section 3, the necessary weak and linearized weak forms are presented and the linearized weak form is discretized using the finite element method. The computational method and algorithms are discussed in Section 4. Finally, Section 5 is devoted to a discussion of the numerical results. In the case of the problem of a plate with an elliptic hole subjected to tension with the applied tension being perpendicular to the major axis, we find (as is to be expected) that the strains are maximum at the vertices along the major axis; however they remain bounded below the value for which the linearization is valid even as the stress increases. In the case of the stepped flat tension bar with shoulder fillets, the strain is maximum at the shoulder but once again remains below the value that guarantees the validity of the linearization.

Unlike the classical linearized model which leads to ever increasing strains that make the model that is being used invalid, the current study is a consistent approach that guarantees that the model that is being used is applicable throughout the domain of application of the model. This fact cannot be overemphasized.

2. Basic equations

2.1. Kinematics

Let $X \in \mathcal{B}$ denote a point in an abstract body $\mathcal{B}$ and $\mathbf{X} = \boldsymbol{\kappa}(X)$ the position of X in the reference configuration $\kappa_r(\mathcal{B})$; we assume there exists a one-to-one function $\boldsymbol{\chi}$ referred to as the motion of the body such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, where $\mathbf{x}$ is the position of X in the current configuration $\kappa_t(\mathcal{B})$ at time t.

The deformation gradient $\mathbf{F}$ and the right and the left Cauchy–Green strain tensors, $\mathbf{C}$ and $\mathbf{B}$, are defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{1}$$

respectively. The displacement field $\mathbf{u}$ is defined through

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{2}$$

Finally, the Green–St. Venant strain ($\mathbf{E}$) and the linearized strain ($\boldsymbol{\varepsilon}$) are defined through

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}), \quad \boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}]. \tag{3}$$

In this work, we consider the case $\|\nabla \mathbf{u}\| \sim O(\delta)$ with $\delta \ll 1$ and thus the relevant strain measure is the linearized strain. Hence, the current and the reference configuration are coincident.

2.2. Equilibrium equation and constitutive relations

In this paper, we study quasi-static problems in the absence of body forces. The equilibrium equation in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$ is

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{0}. \tag{4}$$

For elastic bodies, Rajagopal (2003, 2007) proposed an implicit constitutive relation of the form

$$\boldsymbol{f}(\mathbf{B}, \boldsymbol{\sigma}, \rho) = \mathbf{0}, \tag{5}$$

where ρ is the density of the body. For isotropic bodies, (5) becomes

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2 + \alpha_3\boldsymbol{\sigma} + \alpha_4\boldsymbol{\sigma}^2 + \alpha_5(\mathbf{B}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{B}) + \alpha_6(\mathbf{B}\boldsymbol{\sigma}^2 + \boldsymbol{\sigma}^2\mathbf{B}) + \alpha_7(\mathbf{B}^2\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{B}^2) + \alpha_8(\mathbf{B}^2\boldsymbol{\sigma}^2 + \boldsymbol{\sigma}^2\mathbf{B}^2) = \mathbf{0}, \tag{6}$$

where α_i $(i = 0, 1, 2, \ldots, 8)$ are scalar functions that depend on the invariants

$$\mathrm{tr}\mathbf{B},\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{B}^3,\ \mathrm{tr}\boldsymbol{\sigma},\ \mathrm{tr}\boldsymbol{\sigma}^2,\ \mathrm{tr}\boldsymbol{\sigma}^3,\ \mathrm{tr}(\mathbf{B}\boldsymbol{\sigma}),\ \mathrm{tr}(\mathbf{B}^2\boldsymbol{\sigma}), \mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{B}),\ \mathrm{tr}(\mathbf{B}^2\boldsymbol{\sigma}^2),$$

and the density ρ. For $\|\nabla \mathbf{u}\| \sim O(\delta)$ with $\delta \ll 1$,

$$\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}. \tag{7}$$

On the other hand, using a Taylor expansion in Cartesian coordinates around $\boldsymbol{\varepsilon} = \mathbf{0}$ and assuming that α_i $(i = 0, 1, 2, \ldots, 8)$ does not depend explicitly on ρ, the following approximation holds:

$$\alpha_i(\boldsymbol{\sigma}, \mathbf{B}) \approx \alpha_i(\boldsymbol{\sigma}, \mathbf{I} + 2\boldsymbol{\varepsilon}) \approx \alpha_i(\boldsymbol{\sigma}) + \left.\frac{\partial \alpha_i}{\partial \varepsilon_{kl}}\right|_{(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}=\mathbf{0})} \boldsymbol{\varepsilon} \quad (i = 0, 1, \ldots, 8). \tag{8}$$

On substituting (7) and (8) into (6), the following implicit relation is obtained for terms up to order δ:

$$\aleph_0\mathbf{I} + \aleph_1\boldsymbol{\varepsilon} + \aleph_2\boldsymbol{\sigma} + \aleph_3\boldsymbol{\sigma}^2 + \aleph_4\boldsymbol{\varepsilon}\boldsymbol{\sigma} + \aleph_5\boldsymbol{\sigma}\boldsymbol{\varepsilon} + \aleph_6\boldsymbol{\varepsilon}\boldsymbol{\sigma}^2 + \aleph_7\boldsymbol{\sigma}^2\boldsymbol{\varepsilon} + (\beth_{0_{kl}}\varepsilon_{kl})\mathbf{I} + (\beth_{1_{kl}}\varepsilon_{kl})\boldsymbol{\sigma} + (\beth_{2_{kl}}\varepsilon_{kl})\boldsymbol{\sigma}^2 = \mathbf{0}, \tag{9}$$

where $\aleph_m = \aleph_m(\boldsymbol{\sigma})$ $(m = 0, 1, \ldots, 7)$ and $\beth_{n_{kl}} = \beth_{n_{kl}}(\boldsymbol{\sigma})$ $(n = 0, 1, 2)$ are (in general) nonlinear scalar and tensor functions of the Cauchy stress tensor $\boldsymbol{\sigma}$. Under certain conditions, (9) can be solved for $\boldsymbol{\varepsilon}$. A simple method to find such conditions is the following. On defining the vector $\epsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23})^{\mathrm{T}}$, (9) can be written as the vector equation

$$\mathsf{M}\epsilon = \mathsf{d}, \tag{10}$$

where the matrix $\mathsf{M} = \mathsf{M}_{6\times 6}$ and the vector $\mathsf{d} = \mathsf{d}_{6\times 1}$ depend (in general nonlinearly) on $\boldsymbol{\sigma}$. For brevity, the explicit form of M is not shown here. If $\det \mathsf{M} \neq 0$, then $\epsilon = \mathsf{M}^{-1}\mathsf{d}$ can be computed. For isotropic bodies, (5) can be used to obtain the nonlinear relation (Bustamante and Rajagopal, 2010; Bustamante, 2009)

$$\boldsymbol{\varepsilon} = \boldsymbol{g}(\boldsymbol{\sigma}). \tag{11}$$

For an isotropic body that is described by the classical linearized constitutive relation, the function $\boldsymbol{g}(\boldsymbol{\sigma})$ is

$$\boldsymbol{g}(\boldsymbol{\sigma}) = \frac{1}{E}\boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \tag{12}$$