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Thermal stress analysis of a three-dimensional anticrack in a transversely isotropic solid

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ABSTRACT

A three-dimensional analysis is performed for an infinite transversely isotropic elastic body containing an insulated rigid sheet-like inclusion (an anticrack) in the isotropy plane under a remote perpendicularly uniform heat flow. A general solution scheme is presented for the resulting boundary-value problems. Accurate results are obtained by constructing suitable potential solutions and reducing the thermal problem to a mechanical analog for the corresponding isotropic problem. The governing boundary integral equation for a planar anticrack of arbitrary shape is obtained in terms of a normal stress discontinuity. As an illustration, a complete solution for a rigid circular inclusion is obtained in terms of elementary functions and analyzed. This solution is compared with that corresponding to a penny-shaped crack problem.

1. Introduction

Determining the thermal stresses induced by inhomogeneities is a very important problem in studying the thermoelastic behavior of advanced engineering structures. Knowledge of the thermal stresses is needed to improve the performance of these structures and to predict reliable service lifetimes.

It is well-known that, in addition to cracks, significant stress concentrations occur near the rigid sheet-like edges of inclusions, from which cracking, debonding and damage may emanate. Cracks that are characterized by a displacement discontinuity and flat rigid inclusions (known as anticracks) with a traction discontinuity are two dangerous extremes of inhomogeneities in bodies. Therefore, fully three-dimensional thermoelastic problems involving cracks and anticracks in elastic solids have become the subject of extensive investigations because of the importance of the problems in structural integrity assessments. Considerable advances have been made in studying crack problems with thermal effects: see for example, monographs by Kassir and Sih (1975), Kit and Khay (1989) and Dell'Erba (2002). Solutions of significant problems involving penny-shaped, elliptical and half-infinite plane cracks have been developed by Florence and Goodier (1963), Kassir and Sih (1967), Kassir (1969), Kit and Poberezhnyi (1972), Barber (1975), Krishna Rao and Hasebe (1995), Chaudhuri (2003a,b, 2012) and Stadnyk (2010) for isotropic bodies and by Tsai (1983a,b), Noda and Ashida (1987), Kirilyuk (2001), Podil'chuk (2001), Chen et al. (2004) and Li (2012) for transversely isotropic media. However, considerable fewer studies have been conducted on the thermal stresses around anticracks. Two dimensional problems, such as an insulated or conductive ribbon-like rigid inclusion in an isotropic elastic body, have been studied by Sekine (1977) and Sekine and Mura (1979). Thermoelastic plane problems of the disturbance of a uniform heat by an elliptic rigid inclusion in an anisotropic elastic matrix were investigated by Lin and Hwu (1993) and Chao and Shen (1998). Comparatively fewer three-dimensional analyses have been conducted because of the mathematical difficulties encountered in the solution of these problems. Intractable results for infinite transversely isotropic bodies containing rigid elliptic inclusions under various temperature loads were reported in a review by Podil'chuk (2001). Kaczyński and Kozłowski (2009) developed a method to determine the steady-state thermal stresses and deformations in an elastic isotropic space that has been weakened by an insulated anticrack of arbitrary shape under a uniform perpendicular heat flow. In particular, Kaczyński and Monastyrskyy (2009a) obtained a complete elementary solution for a penny-shaped rigid inclusion with heat conductivity. More recently, Kaczyński and Monastyrskyy (2013) studied a case in which heat flow is incident along the inclusion plane.

For the past several decades, transversely isotropic materials (e.g., hexagonal crystals, some fiber-reinforced composites, piezoelectric materials and rocks) have been widely used in materials science and geomechanics (Ting, 1996). Yue and Selvadurai (1995), Chaudhuri (2003a), Altenbach et al. (2004) and Shodja

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and Ojaghnezhad (2007) have shown that rigid inhomogeneities have a wide range of practical applications. Hence, it is natural to evaluate the strength of transversely isotropic materials using solutions from analogous problems in the theory of thermoelasticity for bodies containing anticracks. The objective of this paper is to include the effect of material anisotropy in analyzing a uniform heat flow that is perpendicularly incident on a rigid lamellar inclusion. Thus, the results obtained by Kaczyński and Kozłowski (2009) are generalized to transversely isotropic materials that are characterized by five elastic constants and two thermal moduli.

This paper is organized as follows. In Section 2, the fundamental equations of linear transversely isotropic thermoelasticity are presented, neglecting the effects of both coupling and inertia. Consequently, a thermal problem and a problem with induced thermal stresses are formulated and solved for an arbitrarily shaped anticrack in Sections 3 and 4, respectively. As an illustration, a complete solution is formulated and analyzed for the penny-shaped rigid inclusion in Section 5. Finally, Section 6 concludes the article.

2. Transversely isotropic uncoupled thermoelasticity

First, we outline the governing equations of thermoelasticity in an uncoupled static setting for a transversely isotropic medium. A more detailed treatment may be found in the monograph by Ding et al. (2006).

The following index notation is used throughout the paper: Latin subscripts always assume values 1, 2 and 3; and Greek subscripts assume values of 1 and 2. The Einstein summation convention holds unless otherwise stated, and subscripts preceded by a comma indicate partial differentiation with respect to the respective coordinates.

In a rectangular Cartesian coordinate system $O\,X_1\,X_2\,X_3$ denote unknown quantities at the point (x_1, x_2, x_3): T denotes the variation in the temperature (where $T = 0$ corresponds to the stress-free state); and the components of the displacement, the stress and the heat flux are denoted by u_i, σ_{ij}, q_i, respectively.

Consider a homogeneous transversely isotropic thermoelastic space and assume that the axis of elastic symmetry coincides with the X_3-axis such that the X_1 and X_2-axes lie in the plane of transverse isotropy.

Neglecting the effect of the strains on the temperature field allows the thermoelastic problem to be re-cast as two separate subproblems that must to be solved consecutively. The first problem is a purely thermal problem that is governed by the Fourier law of steady-state heat conduction and the 3D quasi-Laplace equation for the temperature distribution in the absence of heat sources (Nowinski, 1978)

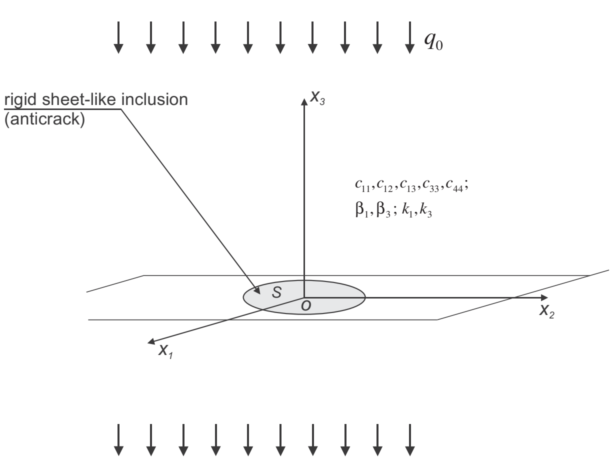

Fig. 1. An anticrack in a transversely isotropic space under a vertically uniform heat flow.

$$q_\alpha = -k_1 T_{,\alpha}, \quad q_3 = -k_3 T_{,3}, \tag{1}$$

$$T_{,\gamma\gamma} + k_0^{-2} T_{,33} = 0, \tag{2}$$

where k_1 and k_3 are the conductivity coefficients in the $O\,X_1\,X_2$ of isotropy plane and the X_3-direction, respectively, and $k_0 = \sqrt{k_1/k_3}$.

After the temperature field has been determined using the prescribed thermal boundary conditions, one can solve the induced thermal stress problem that is governed by the generalized Lamé displacement equations of static equilibrium in the absence of body forces,

$$\begin{aligned} &\frac{1}{2}(c_{11}+c_{12})u_{\gamma,\gamma\alpha} + \frac{1}{2}(c_{11}-c_{12})u_{\alpha,\gamma\gamma} + c_{44}u_{\alpha,33} + (c_{13}+c_{44})u_{3,3\alpha} \\ &\quad = \beta_1 T_{,\alpha}, \quad \alpha = 1,2 \\ &(c_{13}+c_{44})u_{\gamma,\gamma 3} + c_{44}u_{3,\gamma\gamma} + c_{33}u_{3,33} = \beta_3 T_{,3}, \end{aligned} \tag{3}$$

and the constitutive stress-displacements relations for transversely isotropic thermoelastic materials:

$$\sigma_{3\alpha} = c_{44}(u_{\alpha,3} + u_{3,\alpha}), \tag{4}$$

$$\sigma_{33} = c_{13}u_{\gamma,\gamma} + c_{33}u_{3,3} - \beta_3 T, \tag{5}$$

$$\sigma_{12} = \frac{1}{2}(c_{11}-c_{12})(u_{1,2}+u_{2,1}), \tag{6}$$

$$\sigma_{11} = c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} - \beta_1 T, \tag{7}$$

$$\sigma_{22} = c_{12}u_{1,1} + c_{11}u_{2,2} + c_{13}u_{3,3} - \beta_1 T, \tag{8}$$

where the thermal moduli are given as follows:

$$\begin{aligned} \beta_1 &= (c_{11}+c_{12})\alpha_1 + c_{13}\alpha_3, \\ \beta_3 &= 2c_{13}\alpha_1 + c_{33}\alpha_3. \end{aligned} \tag{9}$$

In the equations given above, c_{ij} are the five independent elastic constants, and α_1 and α_3 denote the coefficients of thermal expansion in the isotropy plane and along the X_3-axis, respectively.

3. Thermal anticrack problem and solution

Let us consider a transversely isotropic space that is weakened by a heat-insulated rigid inclusion (anticrack), which occupies a bounded plane area S with a smooth profile in the isotropy plane $x_3 = 0$. There is a constant heat flux $\mathbf{q}(\infty) = [0, 0, -q_0]$, $q_0 > 0$ in the direction of the negative X_3-symmetry axis (Fig. 1). Thus, it is necessary to solve Eq. (2), which satisfies the following thermal boundary conditions:

$$q_3 = -k_3 T_{,3} = 0, \quad (x_1, x_2, x_3 = 0^{\pm}) \in S \quad \text{(heat − insulated inclusion)}, \tag{10}$$

$$q_\alpha = -k_1 T_{,\alpha} = 0, \quad q_3 = -k_3 T_{,3} \to -q_0 \text{ as } \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty \quad \text{(perpendicular heat flux)}. \tag{11}$$

The solution has the following form:

$$T = T^{(0)} + \tilde{T}, \tag{12}$$

where

$$\begin{aligned} &T^{(0)}_{,\gamma\gamma} + k_0^{-2} T^{(0)}_{,33} = 0, \quad (x_1, x_2, x_3) \in R^3, \\ &T^{(0)}_{,1} = T^{(0)}_{,2} = 0, \quad T^{(0)}_{,3} \to \frac{q_0}{k_3} \text{ as } \sqrt{x_1^2 + x_2^2 + x_3^2} \to \infty \end{aligned} \tag{13}$$

and