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On the choice of the reference frame for beam section stiffness properties

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ABSTRACT

This work discusses the choice of a reference frame for beam section stiffness properties. Established concepts as the center of elasticity, the center of stiffness and the center of compliance are discussed and contextualized. An interpretation of univocally defined generalized strain transformations is given in terms of minimization of appropriate norms of the stiffness and compliance matrices of the beam section that univocally define special reference points. Transformations of generalized strain perturbations that preserve the angular strain are sought. They are subsequently constrained to represent a change of reference point, and further restricted to lie in the plane of the section. Each transformation is univocally defined and given a clear mathematical and geometrical interpretation. It is recognized that transformations that decouple forces and linear strains from moments and angular strains cannot be described as a mere change of reference point.

1. Introduction

The notion of 'elastic center' is well present in mechanics. In the second half of the nineteenth century, Karl Culmann developed graphical methods for the design of pile foundations for railroad bridges which involved the notion of elastic center (Culmann, 1866). In 1939, Vetter presents a method based on earlier works of other authors that involves the reduction of forces to an equivalent force applied in the elastic center, which causes a pure translation without rotation, and an equivalent moment which causes a pure rotation about the elastic center (Vetter, 1939). Such problems are extremely simple; they address two-dimensional systems with few rod elements acting along fixed axes; however, they indicate an attention to noteworthy definitions and the choice of points with special properties to find ingenious solutions to engineering problems (Kardestuncer, 1974).

The notions of 'center of stiffness' (CoS) and 'center of compliance' (CoC) have been introduced by Lončarić on solid mathematical foundations for compliant structures using screw theory (Lončarić, 1987), addressing compliant robotic applications. Lipkin et al., based on earlier work (Dimentberg, 1968), discussed the properties of the CoS and CoC, and introduced the 'center of elasticity' (CoE) as the center of the reciprocal three-systems that represent the wrench- and twist-compliant axes of a compliant system (Lipkin and Patterson, 1992; Ciblak and Lipkin, 1994, 1999). Such notions have been extensively used, and are still used nowadays, in several applications ranging from robotics (Roberts, 2002) to biomechanics (Enea et al., 2013). By referring the stiffness of a compliant system to the CoS, forces opposing rotations and moments opposing displacements are maximally decoupled.

In beam theory, the notions of 'shear centroid' (or 'shear center', 'center of twist', 'flexural center', namely the point that must lie along the line of action of a shear force for the section not to twist) and 'axial strain centroid' (or 'tension center', namely the point in a beam section where the neutral axes cross, and where an applied axial load does not produce any bending) are well understood. Nowinski in 1961 discussed an 'axis of twist' and 'center of flexure' for certain classes of anisotropic beams (Nowinski, 1961). Reissner and Tsai discussed the problem for cylindrical shell beams (Reissner and Tsai, 1972). In the seminal work (Giavotto et al., 1983), a simple transformation was proposed to identify the location of the shear and axial strain centroids of the beam section in terms of decoupling linear and angular generalized stresses and strains. However, such procedure cannot be described in terms of a change of reference system. In Rehfield and Atilgan (1989), Kosmatka (1994) and Yu et al. (2002) it is noted that some commonly accepted definitions of characteristic points like the shear center may depend on the spanwise location along the beam, e.g. when bending-torsion coupling is present. In Andreaus and Ruta (1998), a detailed review of the shear center problem is presented. Ecsedi discussed the centre of twist and the centre of shear for

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straight isotropic nonhomogeneous beams (Ecsedi, 2000). Bottasso et al. discussed invariance issues associated with the application of numerical methods, also addressing the case of referring beam sections to arbitrary points (Bottasso et al., 2002). Sapountzakis and Mokos presented an original Boundary Element Method (BEM) solution to transverse shear loading of beams (Sapountzakis and Mokos, 2005) in which transverse loads are applied in the shear center to avoid the induction of twisting moment. The discussion about twist and shear centers is active, as testified by very recent literature on the topic (Barretta, 2012; Ecsedi and Baksa, 2012).

In recent times, the so-called Absolute Nodal Coordinate Formulation (ANCF) became popular also for the analysis of deformable continua, including beams. Apparently, such an approach does not need to care about such issues as the definition of special centroids, since the absolute coordinates of the points that define the geometry of the beam represent the degrees of freedom of the problem, much like for solid nonlinear finite elements. The comparison of ANCF with so-called geometrically exact beam formulations (GEBF) is an active topic of research (Romero, 2008).

This work presents an interpretation of the CoS concept in relation with beam section characterization. Univocally defined generalized strain transformations are interpreted in terms of minimization of appropriate norms of the stiffness matrix of the beam section. To the author's knowledge, such interpretation has never been pointed out before. The beam model is briefly presented in Section 2, focusing on referring linear constitutive properties to an arbitrary reference. The choice of the reference frame for beam section stiffness properties is discussed in Section 3, with a newly proposed definition that specializes Lončarić's CoS to beam stiffness properties. Examples are proposed in Section 4.

2. Beam model

The beam model is formulated using generalized coordinates, namely the position of an arbitrary reference point and the orientation of an arbitrary triad that define the 'pose' of the beam section as a one dimensional Cosserat continuum. See for example the so-called geometrically exact beam formulation named after Reissner-Simo in Ritto-Corrêa and Camotim (2002) and Merlini and Morandini (2013).

The main focus of this work is on the definition of a possibly advantageous frame of reference to express the elastic properties of the beam section, so the choice of a specific approach is deemed inessential, and only the strain energy per unit span of the beam, $\mathscr{W}_{\text{sec}}$, is actually considered.

2.1. Constitutive model

Consider the strain energy per unit span of a beam, $\mathscr{W}_{\text{sec}} = \mathscr{W}_{\text{sec}}(\boldsymbol{\psi})$, where $\boldsymbol{\psi} = \{\boldsymbol{\nu}; \boldsymbol{\kappa}\}$ represents a suitable measure of the generalized strains, namely the linear strain, $\boldsymbol{\nu}$, and the angular strain $\boldsymbol{\kappa}$, as defined, for example, in Ritto-Corrêa and Camotim (2002) and Merlini and Morandini (2013).

The generalized internal forces, namely the internal force, $\mathbf{f}$, and the internal moment, $\mathbf{m}$, are defined as the partial derivatives of the strain energy with respect to the generalized strains, namely

$$\mathbf{f} = \frac{\partial \mathscr{W}_{\text{sec}}}{\partial \boldsymbol{\nu}} \tag{1a}$$

$$\mathbf{m} = \frac{\partial \mathscr{W}_{\text{sec}}}{\partial \boldsymbol{\kappa}} \tag{1b}$$

As a consequence, the internal force and moment are intrinsically expressed with respect to the reference point and orientation of the section, as much as the generalized strains are. In this sense, the stiffness matrix can be seen as the Hessian matrix of the strain energy with respect to the generalized strains; thus,

$$\begin{Bmatrix} \partial \mathbf{f} \\ \partial \mathbf{m} \end{Bmatrix} = \mathbf{K} \begin{Bmatrix} \partial \boldsymbol{\nu} \\ \partial \boldsymbol{\kappa} \end{Bmatrix}, \tag{2}$$

in which $\partial(\cdot)$ indicates a perturbation, following the notation used in Merlini and Morandini (2013). In fact, the constitutive relationship of Eq. (2) must be interpreted as the tangent map that expresses the generalized force increments as functions of the generalized strain increments when beam sections made of hypereleastic material are considered. It applies to generalized finite forces and strains when $\mathbf{K}$ is constant, i.e. when the strains are small (although not necessarily infinitesimal), despite the overall displacements and rotations being arbitrary.

The object of this work is the determination of special reference points for the tangent map between generalized strains and generalized forces. It is worth anticipating that when such map is not constant, those reference points depend on the straining of the beam section, and thus lose their practical appeal, although they preserve a strong mathematical and physical significance. For the sake of simplicity, in the following a stiffness matrix representing a constant tangent map is considered; this fact is taken axiomatically.

In simple models, e.g. those analogous to Conventional Laminate Theory (CLT), the actual inplane straining of the section is implicitly dealt with considering constitutive properties for axial stress state. More sophisticated models, like the one proposed in Giavotto et al. (1983) and subsequent developments (the interested reader may refer to Hodges' book (Hodges, 2006) for more details, and the recent works (Ghiringhelli et al., 2008; Morandini et al., 2010)), explicitly (although often approximately, either axiomatically or in a finite element sense) account for inplane and out-of-plane warping.

The matrix can be partitioned as

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}; \tag{3}$$

submatrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 3×3, with $\mathbf{A}^T = \mathbf{A} > 0$, $\mathbf{C}^T = \mathbf{C} > 0$. The positive definiteness of $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{C}$ can be lost only in degenerate cases that in practice do not need to be considered in this context.

Consider now the corresponding compliance matrix,

$$\mathbf{F} = \mathbf{K}^{-1} = \begin{bmatrix} \overline{\mathbf{A}} & \overline{\mathbf{B}} \\ \overline{\mathbf{B}}^T & \overline{\mathbf{C}} \end{bmatrix} \tag{4}$$

with

$$\overline{\mathbf{A}} = \left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{A}^{-1}, \tag{5a}$$

$$\overline{\mathbf{B}} = -\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{C}^{-1} = -\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1}, \tag{5b}$$

$$\overline{\mathbf{C}} = \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}^T\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{C}^{-1} = \left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)^{-1}. \tag{5c}$$

Later on, it will be used to discuss the reference frame transformation in more detail.

2.2. Change of reference frame

The internal force $\mathbf{f}$ and moment $\mathbf{m}$ can be expressed as functions of the internal force $\mathbf{f}'$ and $\mathbf{m}'$ referred to a different pole, offset by $\mathbf{p}$ from the original reference, and with respect to a different orientation $\mathbf{R}$, both expressed in the reference frame of the section, namely

$$\begin{Bmatrix} \mathbf{f} \\ \mathbf{m} \end{Bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{p} \times \mathbf{R} & \mathbf{R} \end{bmatrix} \begin{Bmatrix} \mathbf{f}' \\ \mathbf{m}' \end{Bmatrix}, \tag{6}$$