International Journal of Solids and Structures 51 (2014) 2448–2459

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

A computational homogenization approach for the yield design of periodic thin plates. Part I: Construction of the macroscopic strength criterion

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ARTICLE INFO

Article history:
Available online 25 March 2014

Keywords:
Yield design
Limit analysis
Homogenization theory
Thin plate model
Second-order cone programming
Finite element method

ABSTRACT

The purpose of this paper is to propose numerical methods to determine the macroscopic bending strength criterion of periodically heterogeneous thin plates in the framework of yield design (or limit analysis) theory. The macroscopic strength criterion of the heterogeneous plate is obtained by solving an auxiliary yield design problem formulated on the unit cell, that is the elementary domain reproducing the plate strength properties by periodicity. In the present work, it is assumed that the plate thickness is small compared to the unit cell characteristic length, so that the unit cell can still be considered as a thin plate itself. Yield design static and kinematic approaches for solving the auxiliary problem are, therefore, formulated with a Love–Kirchhoff plate model. Finite elements consistent with this model are proposed to solve both approaches and it is shown that the corresponding optimization problems belong to the class of second-order cone programming (SOCP), for which very efficient solvers are available. Macroscopic strength criteria are computed for different type of heterogeneous plates (reinforced, perforated plates,...) by comparing the results of the static and the kinematic approaches. Information on the unit cell failure modes can also be obtained by representing the optimal failure mechanisms. In a companion paper, the so-obtained homogenized strength criteria will be used to compute ultimate loads of global plate structures.

1. Introduction

Heterogeneous periodic thin plates are frequently encountered in civil engineering applications and the assessment of their bending strength capacities is of great importance for engineers. The computation of the ultimate load of a structure can be performed using two different class of methods. The first one, called the incremental approach, relies on a step-by-step elasto-plastic computation of the whole loading path until failure. This approach is time consuming, especially for complex structures, and poses convergence issues when approaching the collapse of the structure. The second class concerns direct methods using the theory of limit analysis or, in a more general manner, yield design theory (Salençon, 2013) to bracket the ultimate load using two theorems: the lower bound static approach and the upper bound kinematic approach. The efficiency of direct methods is that they require only the verification of equilibrium equations and the fulfillment of the yield criterion at each point of the structure without any knowledge of the mechanical behavior (apart from the strength criterion) or the whole loading path.

The resolution of the static and kinematic approaches requires solving convex nonlinear optimization problems. Numerical methods dedicated to yield design have gained recent attention due to the development of mathematical programming techniques. A traditional approach involves the linearization of the yield criteria so that the corresponding optimization problems can be formulated within linear programming (Faccioli and Vitiello, 1973; Munro and Da Fonseca, 1978; Sloan, 1988, 1989), for which powerful softwares based on interior point algorithms are available. These algorithms have also been developed for a broader class of optimization problems called second order cone programming (SOCP) (Andersen et al., 1998) and implemented in commercial codes such as the MOSEK software package (Mosek, 2008). Remarkably, a large number of traditional yield criteria can be expressed using conic constraints so that limit analysis problems can be formulated within SOCP (Makrodimopoulos, 2010). Recent works applied this method to 2D plane strain problems (Makrodimopoulos and Martin, 2007; Ciria et al., 2008), frame structures or thin plates

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http://dx.doi.org/10.1016/j.ijsolstr.2014.03.018
0020-7683/

in bending (Le et al., 2010; Bleyer and de Buhan, 2013). The obtained results seem very promising in terms of computational time saving (problems with a large number of optimization variables can be solved within seconds) as well as accuracy. For these reasons, these numerical methods will be used in this work.

Despite the efficiency of the previously mentioned numerical procedures, yield design of periodic media can be very difficult to perform due to the presence of rapidly and strongly varying material properties on a large scale structure. Numerical computations on the heterogeneous structure are, therefore, out of reach due to the high degree of local refinement needed to correctly capture the properties of the inhomogeneities. Homogenization theory in yield design has therefore been developed to characterize the strength domain of an equivalent homogeneous media with the idea that the corresponding homogenized yield design problem would, then, be much easier to solve. Founding works are due to Suquet (1985) and de Buhan (1986).

The determination of the homogenized or macroscopic strength properties are quite similar to homogenization theory in elasticity. The macroscopic strength domain is, indeed, determined by solving an auxiliary yield design problem formulated on the unit cell of the periodic plate. Homogenization of elastic in-plane periodic plates has been widely studied by different authors. One main feature of this problem is that a plate model is valid in the limit of a small thickness h compared to the typical length L of the plate structure whereas the homogenization procedure is valid in the limit of a small in-plane typical length a of the unit cell compared to the same length L. Therefore, different homogenization procedures have to be considered depending on the relative values of a and h.

For instance, if a and h are of the same order, the unit cell has to be modeled as a 3D medium (Bourgeois, 1997; Sab, 2003; Dallot and Sab, 2008). On the contrary, if h is small compared to a, it is possible to replace the original 3D heterogeneous body by a 2D heterogeneous Love–Kirchhoff plate, which is homogenized in a second step (see Duvaut and Metellus (1976) for elastic plates). Our work will be focused on this specific case within the framework of yield design theory and associated numerical methods.

This paper is organized as follows: in Section 2, the homogenization theory in yield design will be briefly described within the framework of thin plates in bending and the equilibrium equations of the associated Love–Kirchhoff plate model will be recalled. Section 3 is devoted to the formulation of the auxiliary yield design problem, either by the static approach or by the kinematic approach, and the definition of the macroscopic strength criterion G^{hom} is given. Numerical methods to solve the static approach are then presented in Section 4 and the corresponding optimization problem is formulated as a SOCP problem. Section 5 deals with the case of the kinematic approach in the same manner. Finally, different numerical examples are studied in Section 6, in order to assess the performance of both numerical procedures.

2. Yield design of periodic thin plates: a homogenization approach

2.1. The heterogeneous problem

We consider a heterogeneous thin plate occupying a domain Ω in the (x, y)-plane. Internal forces of the plate are the tensor of membrane forces $\underline{\underline{N}}$, the tensor of bending moments $\underline{\underline{M}}$ and the vector of shear forces $\underline{V}$. Generally speaking, the set of admissible internal forces with respect to the local strength of the plate at a point $\underline{x} \in \Omega$ can be represented as a bounded convex set $G(\underline{x})$:

$$(\underline{\underline{N}}, \underline{\underline{M}}, \underline{V}) \in G(\underline{x})$$

In the special case of thin plates in bending, it is generally assumed that the plate is *infinitely resistant to both membrane and shear forces* such that the local strength criterion depends on the bending moment only:

$$\underline{\underline{M}} \in G(\underline{x})$$

Now, assuming that the plate loading depends upon several loading parameters $\underline{Q}$, the domain Λ of potentially safe loads $\underline{Q}$ is defined as the set of loads such that there exists a statically admissible (S.A.) bending moment field $\underline{\underline{M}}(\underline{x})$ (i.e. which equilibrates the loading $\underline{Q}$), satisfying the strength criterion at each point of the plate (see Salençon, 2013):

$$\Lambda = \left\{ \underline{Q} \mid \exists \underline{\underline{M}}(\underline{x}) \text{ S.A. with } \underline{Q},\ \forall \underline{x} \in \Omega \quad \underline{\underline{M}}(\underline{x}) \in G(\underline{x}) \right\}$$

Making use of the virtual work principle, one can obtain a kinematic definition of Λ, dual to the previous static one. In the case of thin plates in bending, the hypothesis of infinite membrane and shear strength imposes that the plate kinematics obey the Love–Kirchhoff condition. Let $\hat{u}$ be the virtual transversal velocity of the plate and $\underline{q}$ the generalized kinematic parameters defined by duality in the expression of the work of external forces, such that for all $\hat{u}$ kinematically admissible (K.A.) with $\underline{q}$ (i.e. piecewise continuous and differentiable satisfying the kinematic boundary conditions), the virtual work of external load is given by $P_{ext}(\hat{u}) = \underline{Q} \cdot \underline{q}$. We then introduce $\pi(\hat{\underline{\underline{\chi}}}; \underline{x})$, the support function of $G(\underline{x})$ defined as

$$\pi(\hat{\underline{\underline{\chi}}}; \underline{x}) = \sup_{\underline{\underline{M}} \in G(\underline{x})} \underline{\underline{M}} : \hat{\underline{\underline{\chi}}}$$

and the associated maximum resisting work $P_{rm}(\hat{u})$ as follows[1]:

$$P_{rm}(\hat{u}) = \int_\Omega \pi(\hat{\underline{\underline{\chi}}}; \underline{x}) d\Omega$$

where $\hat{\underline{\underline{\chi}}} = \underline{\underline{\nabla_s \nabla \hat{u}(\underline{x})}}$ is the curvature tensor associated with the virtual velocity field $\hat{u}$. The following kinematic definition of Λ is then obtained:

$$\underline{Q} \in \Lambda \Rightarrow \forall \hat{u} \text{ K.A. with } \underline{q}, \quad P_{ext}(\hat{u}) \leqslant P_{rm}(\hat{u})$$

2.2. The homogenized problem

Now, the special case of plates which are periodic in their in-plane direction will be considered. Therefore, there exist two vectors $\underline{a}_1, \underline{a}_2$ such that $G(\underline{x})$ can be reproduced by periodicity along $\underline{a}_1$ and $\underline{a}_2$:

$$G(\underline{x} + n_1\underline{a}_1 + n_2\underline{a}_2) = G(\underline{x}) \quad \forall \underline{x} \in \Omega,\ \forall n_1, n_2 \in \mathbb{Z}$$

The two vectors $\underline{a}_1$ and $\underline{a}_2$ define the unit cell of the periodic plate. In the case when the typical size a of the unit cell is small in comparison to the plate typical length L ($a \ll L$), the natural idea of homogenization theory is to substitute the local heterogeneous strength criterion $G(\underline{x})$ by a homogenized or macroscopic strength criterion G^{hom}, as illustrated in Fig. 1.

Using the same definitions as before, we introduce

$$\Lambda^{hom} = \left\{ \underline{Q} \mid \exists \underline{\underline{M}}(\underline{x}) \text{ S.A. with } \underline{Q},\ \forall \underline{x} \in \Omega \quad \underline{\underline{M}}(\underline{x}) \in G^{hom} \right\}$$

which also admits the following kinematic definition:

$$\underline{Q} \in \Lambda^{hom} \Rightarrow \forall \hat{u} \text{ K.A. with } \underline{q} \quad P_{ext}(\hat{u}) \leqslant P^{hom}_{rm}(\hat{u}) = \int_\Omega \Pi_{hom}(\hat{\underline{\underline{\chi}}}) d\Omega$$

where $\Pi_{hom}(\hat{\underline{\underline{\chi}}})$ is the support function associated with G^{hom}.

[1] This expression assumes that the rotation vector associated with the gradient of the transversal velocity field $\hat{u}$ is everywhere continuous. If this is not the case, another term taking into account the contribution of angular jumps has to be considered in the expression of the maximum resisting work. For more details, we refer to Bleyer and de Buhan (2013) and Section 5.